EXISTENCE OF SOLUTIONS FOR GENERALIZED QUASIEQUILIBRIUM PROBLEMS

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ABSTRACT

In this paper, we establish some existence theorems by using Kakutani-Fan-Glicksberg fixed-point theorem for generalized quasiequilibrium problems in real locally convex Hausdorff topological vector spaces. Moreover, we also discuss closeness of the solution sets of generalized quasiequilibrium problems. The results presented in the paper improve and extend the main results of Long et al in [3], Plubtieng - Sitthithakerngkietet in [5] and Yang-Pu in [6].

Keywords: ceneralized quasiequilibrium problems, Kakutani-Fan-Glicksberg fixed-point theorem, closeness.

TÓM TẮT

Sự tồn tại nghiệm cho bài toán tựa cân bằng tổng quát

Trong bài báo này, chúng tôi thiết lập một số định lí tồn tại nghiệm bằng cách sử dùng định lí điểm bất động Kakutani-Fan-Glicksberg cho bài toán tựa cân bằng tổng quát trong không gian tôpô Hausdorff thực lồi địa phương. Ngoài ra, chúng tôi cũng thảo luận tính đóng của các tập nghiệm của bài toán tựa cân bằng tổng quát. Kết quả trong bài báo là cải thiện và mở rộng các kết quả chính của Long và các tác giả trong [3], Plubtieng - Sitthithakerngkietet trong [5] và Yang-Pu trong [6].

Từ khóa: các bài toán tựa cân bằng tổng quát, định lí điểm bất động Kakutani-Fan-Glicksberg, tính đóng của tập nghiệm.

1. Introduction and Preliminaries

Let X, Y, Z be real locally convex Hausdorff topological vector spaces, $A \subseteq X$ and $B \subseteq Y$ be nonempty compact convex subsets and $C \subseteq Z$ is a nonempty closed compact convex cone. Let $K_1 : A \to 2^A$, $K_2 : A \to 2^A$, $T : A \to 2^B$ and $F : A \times B \times A \to 2^Z$ be multifunctions.

We consider the following generalized quasiequilibrium problems (in short, (QEP_1) and (QEP_2)), respectively:

(QEP_1): Find $\bar{x} \in A$ such that $\bar{x} \in K_1(\bar{x})$ and $\exists \bar{z} \in T(\bar{x})$ satisfy

$$F(\bar{x}, \bar{z}, y) \subset C, \forall y \in K_2(\bar{x})$$

and

(QEP_2): $\bar{x} \in A$ such that $\bar{x} \in K_1(\bar{x})$ and $\forall \bar{z} \in T(\bar{x})$ satisfying

$$F(\bar{x}, \bar{z}, y) \subset C, \forall y \in K_2(\bar{x}).$$

We denote that $S_1(F)$ and $S_2(F)$ are the solution sets of (QEP_1) and (QEP_2), respectively.

If $K_1 = K_2 = K$, then (QEP_1) becomes strong vector quasiequilibrium problem (in short,(QEP)). This problem has been studied in [3, 5].

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(QEP): Find $\bar{x} \in A$ and $\bar{z} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$F(\bar{x}, \bar{z}, y) \subset C,$ for all $y \in K(\bar{x}).$

If $K_1(x) = K_2(x) = K(x), T(x) = \{z\}$ for each $\bar{x} \in A$, then (QEP$_1$) becomes strong vector equilibrium problem (in short,(EP)). This problem has been studied in [6].

(EP): Find $\bar{x} \in A$ such that $\bar{x} \in K(\bar{x})$ and

$F(\bar{x}, y) \subset C,$ for all $y \in K(\bar{x}).$

The structure of our paper is as follows. In the remaining part of this section we recall definitions for later uses. In Section 2, we establish some existence and closeness theorems by using Kakutani-Fan-Glicksberg fixed-point theorem for generalized quasiequilibrium problems with set-valued mappings in real locally convex Hausdorff topological vector spaces.

Now we recall some notions in [1, 2, 4]. Let X and Z be as above and $G: X \to 2^Z$ be a multifunction. G is said to be lower semicontinuous (lsc) at x_0 if $G(x_0) \cap U \neq \varnothing$ for some open set $U \subseteq Z$ implies the existence of a neighborhood N of x_0 such that, for all $x \in N, G(x) \cap U \neq \varnothing$. An equivalent formulation is that: G is lsc at x_0 if $\forall x_\alpha \to x_0$, $\forall z_0 \in G(x_0), \exists z_\alpha \in G(x_\alpha), z_\alpha \to z_0$. G is called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq G(N)$. Q is said to be Hausdorff upper semicontinuous (H-usc in short; Hausdorff lower semicontinuous, H-lsc, respectively) at x_0 if for each neighborhood B of the origin in Z, there exists a neighborhood N of x_0 such that, $Q(x) \subseteq Q(x_0) + B, \forall x \in N$ ($Q(x_0) \subseteq Q(x) + B, \forall x \in N$). G is said to be continuous at x_0 if it is both lsc and usc at x_0 and to be H-continuous at x_0 if it is both H-lsc and H-usc at x_0. G is called closed at x_0 if for each net $\{(x_\alpha, z_\alpha)\} \subseteq \text{graph}G: = \{(x,z) \mid z \in G(x)\}, (x_\alpha, z_\alpha) \to (x_0, z_0)$, z_0 must belong to $G(x_0)$. The closeness is closely related to the upper (and Hausdorff upper) semicontinuity. We say that G satisfies a certain property in a subset $A \subseteq X$ if G satisfies it at every points of A. If $A = X$ we omit "in X" in the statement.

Lemma 1.1. *([2], [4])*

Let X and Z be two Hausdorff topological spaces and A be a nonempty subset of X and $F: A \to 2^Z$ be a multifunction. If F has compact values, then F is usc at x_0 if and only if for each net $\{x_\alpha\} \subseteq A$ which converges to x_0 and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there are $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y.$

Definition 1.2. *([4])*

Let X, Y be two topological vector spaces and A be a nonempty subset of X and let $F: A \to 2^Y$ be a set-valued mapping, with $C \subset Y$ is a nonempty closed compact convex cone.

(i) F is called *upper* C*-continuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that, for all $x \in V$,

$$F(x) \subseteq F(x_0) + U + C, \forall x \in V.$$

(ii) F is called *lower* C*-continuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that, for all $x \in V$,

$$F(x_0) \subseteq F(x) + U - C, \forall x \in V.$$

Lemma 1.3. *([4])*

Let X and Y be two Hausdorff topological spaces and $F : X \to 2^Y$ be a set-valued mapping.

(i) If F is upper semicontinuous with closed values, then F is closed;

(ii) If F is closed and Y is compact, then F is upper semicontinuous.

Lemma 1.4.

(Kakutani-Fan-Glickcberg (See [2, 4])).

Let A be a nonempty compact subset of a locally convex Hausdorff vector topological space Y. If $M : A \to 2^A$ is upper semicontinuous and for any $x \in A, M(x)$ is nonempty, convex and closed, then there exists an $x^* \in A$ such that $x^* \in M(x^*)$.

2. Existence of solutions

In this section, we discuss existence and closeness of the solution set of generalized quasiequilibrium problems by using Kakutani-Fan-Glicksberg fixed-point theorem.

Definition 2.1.

Let X and Z be two Hausdorff topological spaces and A be a nonempty subset of X and $C \subset Z$ is a nonempty closed compact convex cone. Suppose $F : A \to 2^Z$ be a multifunction. F is said to be *generalized* C*-quasiconvex at* $x_0 \in A$, if $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$ such that $F(x_1) \subset C$ and $F(x_2) \subset C$, we have

$$F(\lambda x_1 + (1-\lambda)x_2) \subset C.$$

Remrk 2.2.

To see the nature of the above quasiconvexity, let us consider the simplest case when A= X = Z = R, $F : R \to R$ is single-valued and $C = R_-$. Then $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$, if $F(x_1) \le 0, F(x_2) \le 0$, then

$F((1-\lambda)x_1 + \lambda x_2)) \le 0$. This means that F is modified 0-level quasiconvex, since the classical quasiconvexity says that $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$,

$$F((1-\lambda)x_1 + \lambda x_2)) \le \max\{F(x_1, F(x_2)\}.$$

Theorem 2.3.

Assume for (QEP_1) that

(i) K_1 is upper semicontinuous in A with nonempty convex closed values and K_2 is lower semicontinuous in A with nonempty closed values;

(ii) T is upper semicontinuous in A with nonempty convex compact values;

(iii) for all $(x,z) \in A \times B$, $F(x,z,K_2(x)) \subset C$;

(iv) for all $(z,y) \in B \times A$, $F(.,z,y)$ is generalized C-quasiconvex;

(v) F is upper C-continuous.

Then, (QEP$_1$) has a solution. Moreover, the solution set of (QEP$_1$) is closed.

Proof.

For all $(x,z)\in A\times B$, define a set-valued mapping: $\Psi: A\times B\to 2^A$ by

$$\Psi(x,z)=\{t\in K_1(x): F(t,z,y)\subset C, \forall y\in K_2(x)\}.$$

Step 1. We show that $\Psi(x,z)$ is nonempty.

Indeed, for all $(x,z)\in A\times B$, $K_1(x), K_2(x)$ are nonempty. Thus, by assumption (iii), we have $\Psi(x,z)\neq\varnothing$.

Step 2. We show that $\Psi(x,z)$ is convex subset of A.

Let $t_1,t_2\in\Psi(x,z)$, $\alpha\in[0,1]$ and put $t=\alpha t_1+(1-\alpha)t_2$. Since $t_1,t_2\in K_1(x)$ and $K_1(x)$ is convex set, we have $t\in K_1$. Thus, for $t_1,t_2\in\Psi(x,z)$, it follows that

$$F(t_i,z,y)\subset C, i=1,2, \forall y\in K_2(x).$$

By (iv), F(., z, y) is generalized C-quasiconvex

$$F(\alpha t_1+(1-\alpha)t_2,z,y)\subset C, \forall\alpha\in[0,1], \forall y\in K_2(x),$$

i.e., $t\in\Psi(x,z)$. Therefore, $\Psi(x,z)$ is a convex subset of A.

Step 3. We show that $\Psi(x,z)$ is upper semicontinuous with nonempty closed convex values. Since A is compact, by Lemma 1.3 (ii), we need only show that Ψ is a closed mapping. Indeed, let a net $\{(x_n,z_n)\}\subseteq A\times B$ such that $(x_n,z_n)\to(x,z)\in A\times B$, and let $t_n\in\Psi(x_n,z_n)$ such that $t_n\to t_0$. We now need to show that $t_0\in\Psi(x,z)$. Since $t_n\in K_1(x_n)$ and K_1 is upper semicontinuous with nonempty closed values, hence K_1 is closed, thus we have $t_0\in K_1(x)$. Suppose to the contrary of $t_0\notin\Psi(x,z)$. Then, $\exists y_0\in K_2(x)$ such that

$$F(t_0,z,y_0)\not\subset C,$$

Which implies that there exists a neighborhood U of the origin in Z, such that

$$F(t_0,z,y)+U\not\subset C.$$

By condition (v), for any neighborhood U_1 of 0 in Z, there exists a neighborhood $V(t_0,z,y)$ of (t_0,z,y) such that

$$F(t_0',z',y')\subset F(t_0,z,y)+U_1+C, \forall(t_0',z',y')\in V(t_0,z,y).$$

Without loss of generality, we can assume that $U_1=U$. This implies that

$$F(t_0',z',y')\subset F(t_0,z,y)+U_1+C\not\subset C+C\subset C,$$
$$\forall(t_0',z',y')\in V(t_0,z,y).$$

Thus there is $n_0\in I$ such that

$$F(t_n,z_n,y_n)\not\subset C, \forall n\geq n_0,$$

which contradicts to $t_n\in\Psi(x_n,z_n)$. Thus, $t_0\in\Psi(x,z)$.

Step 4. Now we need to prove the solutions set $S_1(F)\neq\varnothing$.

Define the set-valued mapping $H: A\times B:\to 2^{A\times B}$ by

$$H(x,z)=(\Psi(x,z),T(x)), \forall(x,z)\in A\times B.$$

Then H is upper semicontinuous and $\forall(x,z)\in A\times B, H(x,z)$ is a nonempty closed convex subset of $A\times B$. By Lemma 1.4, there exists a point $(x^*,z^*)\in A\times B$ such that $(x^*,z^*)\in H(x^*,z^*)$, that is $x^*\in\Psi(x^*,z^*)$, $z^*\in T(x^*)$, which implies that there exists $x^* \in A$ and $z^* \in T(x^*)$ such that $x^* \in K_1(x^*)$ and $F(x^*, z^*, y) \subset C$,

i.e., $x^* \in S_1(F)$.

Step 5. Now we prove that $S_1(F)$ is closed. Indeed, let a net $\{x_n, n \in I\} \subset S_1(F)$: $x_n \to x_0$. As $x_n \in S_1(F)$, there exists $z_n \in T(x_n)$ such that

$$F(x_n, z_n, y) \subset C, \forall y \in K_2(x_n).$$

Since K_1 is upper semicontinuous with nonempty closed values, hence K_1 is closed. Thus, $x_0 \in K_1(x_0)$. Since T is upper semicontinuous with nonempty compact values, then T is closed, hence we have $z \in T(x_0)$ such that $z_n \to z$. By the condition (v), we have

$$F(x_0, z, y) \subset C, \forall y \in K_2(x_0).$$

This means that $x_0 \in S_1(F)$. Thus $S_1(F)$ is closed. □

In the special case $K_1 = K_2 = K$, we have the following Corollary.

Corollary 2.4.

Assume for (QEP) that

(i) K is continuous in A with nonempty closed convex values;

(ii) T is upper semicontinuous in A with nonempty compact convex values;

(iii) for all $(x,z) \in A \times B$, $F(x, z, K_2(x)) \subset C$;

(iv) for all $(z, y) \in B \times A$, $F(., z, y)$ is generalized C-quasiconvex;

(v) F is upper C-continuous;

Then, (QEP) has a solution. Moreover, the solution set of (QEP) is closed.

Proof. The result is derived from the technics of the proof for Theorem 2.3. □

Remark 2.5.

In the special case as above, Corollary 2.4 reduces to Theorem 3.1 in [3].

However, our Corollary 2.4 is stronger than Theorem 3.1 in [3]. The following example shows that in this special case, all assumptions of Corollary 2.4 are satisfied. However, Theorem 3.1 in [3] is not fulfilled.

Example 2.6.

Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = [0,4]$ and let $K_1(x) = K_2(x) = [0,1]$ and $T_1(x) = T_2(x) = [\frac{1}{5}, 1]$

$$F(x,z,y) = \begin{cases} [\frac{1}{2}, 1] & \text{if } x_0 = y_0 = z_0 = \frac{1}{2}, \\ [1,2] & \textit{otherwise}. \end{cases}$$

We see that all assumptions of Corollary 2.4 are satisfied. So by this corollary the considered problem has solutions. However, F is not lower $(-C)$-continuous at $x_0 = \frac{1}{2}$. Also, Theorem 3.1 in [3] does not work.

Corollary 2.7.

Assume for (EP) that

(i) S is continuous in A with nonempty convex closed values;

(ii) for all $x \in A$, $F(x, K(x)) \subset C$;

(iii) for all $y \in A$, $F(., y)$ is strongly C-quasiconvex;

(iv) the set F is upper C-continuous.

Then, (EP) has a solution. Moreover, the solution set of (EP) is closed.

Proof. The result is derived from the technics of the proof for Theorem 2.3.

Remark 2.8.

In the special case as above, Corollary 2.4 and Corollary 2.7 reduces to Theorem 3.1 in [3] and Theorem 3.3 in [6], respectively. However, our Corollary 2.4 and Corollary 2.7 is stronger than Theorem 3.1 in [3] and Theorem 3.3 in [6]. The following example shows that the assumption generalized C-quasiconvex of Corollary 2.4 and Corollary 2.7 is satisfied, but the assumption C-quasiconvex of Theorem 3.1 in [3] and Theorem 3.3 in [6] is not fulfilled.

Example 2.9.

Let $A, B, X, Y, Z, K_1, K_2, C$ as in Example 2.6 and $T(x) = [0,1]$ and

$$F(x,z,y) = \begin{cases} [1,2] & \text{if } x_0 = y_0 = z_0 = \frac{1}{2}, \\ [\frac{1}{2},1] & \textit{otherwise}. \end{cases}$$

We see that the assumption generalized C-quasiconvex is satisfied. However, F is not C-quasiconvex at $x_0 = \frac{1}{2}$.

Passing to the problem (QEP$_2$), we also have the following similar results as that of Theorems 2.3.

Theorem 2.10.

Assume for (QEP$_2$) that

(i) K_1 is upper semicontinuous in A with nonempty closed convex values and K_2 is lower semicontinuous A with nonempty closed values;

(ii) T is lower semicontinuous in A with nonempty convex values;

(iii) for all $(x,z) \in A \times B$, $F(x,z,K_2(x)) \subset C$;

(iv) for all $(z,y) \in B \times A$, $F(.,z,y)$ is generalized C-quasiconvex;

(v) F is upper C-continuous;

Then, (QEP$_2$) has a solution. Moreover, the solution set of (QEP$_2$) is closed.

Proof.

We omit the proof since the technique is similar as that for Theorem 2.3 with suitable modifications. □

Remark 2.11.

Note that, if we let X, Y, Z be real locally G - convex Hausdorff topological vector spaces, then, the results in this paper is extended the results of Plubtieng - Sitthithakerngkietet in [5] as in Remark 2.5, Example 2.6 and 2.9.

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(Received: 20/02/2012; Accepted: 24/4/2012)