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CLASSIFICATION OF φ -MAXIMAL SPACELIKE AFFINE TRANSLATION SURFACES IN THE MINKOVSKI SPACE $\mathbb{R}^3_1$ WITH DENSITY e^z

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ABSTRACT

An affine translation surface is a graph of a function $z(x,y) = f(x) + g(y + ax)$, $a \neq 0$, *introduced by Liu and Yu in 2013. The article considers the spacelike affine translation surfaces in the Minkowski space* $\mathbb{R}^3_1$ *with density* e^z, *establishing the Lagrange's equation type for* φ *-maximal surface, classifying* φ *-maximal spacelike affine translation surfaces. The result obtains two parameters (2.3) and (2.4). From that, the Calabi – Bernstein theorem in this space is not true because two function* f *and* g *are defined on* $\mathbb{R}$.

Keywords: φ -maximal, density, affine translation surface, Lagrange's equation.

TÓM TẮT

Phân loại các mặt tịnh tiến afin kiểu không gian φ -cực đại trong không gian Minkovski $\mathbb{R}^3_1$ với mật độ e^z

Mặt tịnh tiến afin là đồ thị của hàm $z(x,y) = f(x) + g(y + ax)$, $a \neq 0$, *được Liu và Yu giới thiệu vào năm 2013. Bài báo xét các mặt tịnh tiến afin kiểu không gian trong không gian Minkovski* $\mathbb{R}^3_1$ *với mật độ* e^z, *thiết lập phương trình kiểu Lagrange cho mặt* φ *-cực đại và phân loại các mặt tịnh tiến afin kiểu không gian* φ *-cực đại. Kết quả thu được hai họ tham số (2.3) và (2.4). Qua đó, định lí Calabi - Bernstein trong không gian đang xét là không đúng do hai hàm* f *và* g *trong các tham số xác định trên toàn bộ* $\mathbb{R}$.

Từ khóa: φ -cực đại, mật độ, mặt tịnh tiến afin, phương trình Lagrange.

1. Introduction

Manifolds with density, a new category in geometry, is attracting many Mathematicians, appeared naturally in mathematics and physics. A manifold with density is a Riemannian manifold (M^n, g) with a positive density function e^{φ} used to weight all k -

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dimensional volumes, $1 \le k \le n$ [1]. On this manifold, φ -mean curvature H_φ of a hypersurface Σ is given

$$H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{n}}, \tag{1.1}$$

Where H is the Euclidean mean curvature and $\mathbf{n}$ is the normal vector field of the hypersurface Σ. The expression (1.1) is obtained by the first variation of φ -area [2], [3]. A hypersurface with $H_\varphi = 0$ is called φ -minimal. If the Riemannian manifold M is alternated by a Minkowski space, the expression (1.1) is also satisfied the variation of φ -area [4], [5]. Then, a hypersurface $H_\varphi = 0$ is called φ -maximal.

The literature of minimal surfaces in Euclidean spaces began with Lagrange in 1760. Finding a critical point for the functional area gives a second order differential equation, called the Lagrange's equation. At that time, the only known solution of Lagrange's equation was linear functions. In 1835, Scherk solved Lagrange's equation for translation functions, i.e. functions of the type $z(x,y) = f(x) + g(y)$ and discovered the Scherk's minimal surfaces. Recently, Liu and Yu has defined the affine translation surfaces that are the graphs of functions $z(x,y) = f(x) + g(y + ax), a \neq 0$, classified all the minimal surfaces [6]. After that, Liu and Jung classified all the affine translation surface with constant mean curvature [7].

In the space $\mathbb{R}^3$ with density e^z, Hieu and Hoang classifies all ruled φ -minimal surfaces and that all translation minimal surfaces are ruled [8]. He and Zhao classifies all weighted minimal surfaces [9]. Recently, some authors promoted the research the weighted maximal surface in the Minkowski space with density and the weighted minimal surface in the Calilean space with density [4], [10].

If the Euclidean space $\mathbb{R}^3$ is alternated by the Minkowski space $\mathbb{R}^3_1$, a surface of constant mean curvature $H = 0$ is called a *maximal surface*. The classified result of all spacelike maximal translate surfaces in $\mathbb{R}^3_1$ is the planes and the Sherk surfaces [11]. Recently, Yan and Liu contributed some maximal affine translate [12]. Motivated by the nice work mentioned above, we classified the φ -maximal affine translate surfaces in the space $\mathbb{R}^3_1$ with density e^z [5]. In this article, we established the Lagrange's equation type for the -maximal affine translate surfaces in the space $\mathbb{R}^3_1$ with density e^z. Section 2.2, by a different technique in the other articles, proved that the function f_u or g_v must be constant, classified the spacelike φ -maximal affine translate surfaces. The result shows that two functions f, g are defined on entire $\mathbb{R}$. Since, the Calabi-Bernstein theorem in this space is not true. Section 2.3 presents some images to illustrate the article's result.

2. Main contents

2.1. The affine translate surface in the Minkowski space

2.1.1. The Minkowski space

The Minkowski space, denote $\mathbb{R}^3_1$, is the real vector space $\mathbb{R}^3$ with non-degenerate symmetric bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

where $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Since $\langle , \rangle$ is not positive definite, $\langle \mathbf{x}, \mathbf{x} \rangle$ can be equal 0 or negative. A vector $\mathbf{x} \in \mathbb{R}^3_1 \setminus \{(0,0,0)\}$ is called *spacelike* (resp. *lightlike* or *timelike*) if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ (resp. $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$). The n*orm* of vector $\mathbf{x}$, denote $\|\mathbf{x}\|$, is the positive number $\sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. The *Lorentzian vector product* of $\mathbf{x}$ and $\mathbf{y}$, denote $\mathbf{x} \times \mathbf{y}$, is defined by

$$\mathbf{x} \times \mathbf{y} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, -\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$

A regular surface $\Sigma \subset \mathbb{R}^3_1$ is called *spacelike* if the $\langle , \rangle|_\Sigma$ is positive definite. It means that all tangent vectors of Σ are spacelike. If E, F, G and L, M, N are coefficients of the first and second fundamental form, respectively, the mean curvature H of Σ is given by

$$H = \frac{-1}{2} \frac{LG - 2MF + NE}{EG - F^2}.$$

For more details about the Minkowski space, we refer the reader to [11].

2.1.2. The affine translate surface

The affine translate surface (S) in the Minkowski space $\mathbb{R}^3_1$ is defined as a parameter surface in $\mathbf{R}^3$ which can be written as

$$X(x, y) = \big(x, y, f(x) + g(y + ax)\big), x, y \in I \subset \mathbb{R}, \tag{1.2}$$

For some nonzero constant a and differential functions f and g.

By a direct calculation, the coefficients of the first fundamental form of (S) is given by

$$E = 1 - (f' + ag')^2, F = -g'(f' + ag'), G = 1 - g'^2,$$

where $f' = \dfrac{df(x)}{dx}$ và $g' = \dfrac{dg(y+ax)}{d(y+ax)}$.

Therefore, $EG - F^2 = 1 - (f' + ag')^2 - g'^2$.

Then, (S) is a spacelike surface if and only if

$$1 - (f' + ag')^2 - g'^2 > 0. \qquad (1.3)$$

The unit normal vector and the coefficients of the second fundamental form of (S) is given by

$$\mathbf{n} = \frac{-1}{\sqrt{1-(f'+ag')^2-g'^2}}(f'+ag', g', 1),$$

$$L = \frac{f'' + a^2 g''}{\sqrt{1-(f'+ag')^2-g'^2}},$$

$$M = \frac{ag''}{\sqrt{1-(f'+ag')^2-g'^2}},$$

$$N = \frac{g''}{\sqrt{1-(f'+ag')^2-g'^2}}.$$

Therefore,

$$H = \frac{-1}{2}\frac{(f''+a^2g'')(1-g'^2) - 2[-g'(f'+ag')]ag'' + [1-(f'+ag')^2]g''}{\left[1-(f'+ag')^2-g'^2\right]^{3/2}}$$

$$= \frac{-1}{2}\frac{f''(1-g'^2) + (1+a^2-f'^2)g''}{\left[1-(f'+ag')^2-g'^2\right]^{3/2}}.$$

2.2. Classification of φ-maximal spacelike affine translation surfaces in the Minkovski space $\mathbb{R}^3_1$ with density e^z

2.2.1. The φ-maximal surface

Consider the Minkowski space $\mathbb{R}^3_1$ with density e^{φ} used to weight both volume and perimeter area. In terms of the underlying Riemannian volume dV_{φ} and area dA_{φ}, the new, weighted volume and area are given by

$$dV_{\varphi} = e^{\varphi} dV \text{ và } dA_{\varphi} = e^{\varphi} dA,$$

Then $(\mathbb{R}^3_1, e^{\varphi})$ is called *the Minkovski space* $\mathbb{R}^3_1$ *with density* e^z.

Let Σ be a regular surface in the space $\mathbb{R}^3_1$ with density e^{φ}. *The weighted mean curvature* or *φ-mean curvature* of Σ, denote H_φ, is defined by

$$H_\varphi = H - \frac{1}{2}\frac{d\varphi}{d\mathbf{n}},$$

where H and $\mathbf{n}$ are mean curvature and unit normal vector field of Σ.

Surface Σ is called *φ-maximal* if $H_\varphi = 0$.

2.2.2. Theorem

Let (S) be a spacelike affine translation surface, defined by (1.2), in the Minkowski space $\mathbb{R}^3_1$ with density e^z. Surface (S) is φ-maximal if and only if its parameter is given by

$$X(x,y) = \left(x, y, \left(1 - b^2\right)\ln\cosh\left(\frac{x}{\sqrt{1-b^2}} + c\right) + by + d\right), \tag{1.4}$$

or

$$X(x,y) = \left(x, y, \frac{1+a^2-b^2}{a^2+1}\ln\cosh\left(\frac{ax+y}{\sqrt{1+a^2-b^2}} + c\right) + \frac{x-ay}{a^2+1} + d\right), \tag{1.5}$$

where b, c, d are real constant and $b^2 < 1$.

Remark. Since two parameters are defined (1.4) and (1.5) are defined on entire $\mathbf{R}^2$, The Calabi – Bernstein theorem on the Minkowski space $\mathbb{R}^3_1$ with density e^z is not true.

The proof of Theorem 2.2.2.

We consider parameter (1.2) of surface (S) and function $\varphi(x,y,z) = z$. Then,

$$\frac{d\varphi}{d\mathbf{n}} = \frac{-1}{\sqrt{1-(f' + ag')^2 - g'^2}}.$$

Therefore, surface (S) is φ-maximal if and only if

$$f''(1 - g'^2) + (1 + a^2 - f'^2)g'' - 1 + (f' + ag')^2 + g'^2 = 0. \tag{1.6}$$

We set $u = x$ and $v = y + ax$. Thus, the equation (1.6) can be written as

$$\left(1 - g_v^2\right)f_{uu} + \left(1 + a^2 - f_u^2\right)g_{vv} - 1 + \left(f_u + ag_v\right)^2 + g_v^2 = 0. \tag{1.7}$$

From (1.7), we have

$$g_{vv} = \frac{1 - f_u^2 - f_{uu} - 2af_ug_v - \left(1 + a^2 - f_{uu}\right)g_v^2}{1 + a^2 - f_u^2}. \tag{1.8}$$

Differentiating (1.7) with respect with u, we obtain

$$\left(1-g_v^2\right)f_{uuu}-2f_uf_{uu}g_{vv}+2\left(f_u+ag_v\right)f_{uu}=0. \tag{1.9}$$

Substituting (1.8) into (1.9), we get

$$\begin{aligned}&\left[\left(1+a^2-f_u^2\right)f_{uuu}+2f_uf_{uu}\left(1+a^2-f_{uu}\right)\right]g_v^2+\\&+2af_{uu}\left(1+a^2+f_u^2\right)g_v+\left(1+a^2-f_u^2\right)f_{uuu}+2f_uf_{uu}\left(a^2+f_{uu}\right)=0.\end{aligned} \tag{1.10}$$

Then (1.10) can be regarded as a polynomial equation respect to g_v. We distinguish two cases.

+ *Assume that* g_v *is a constant*, i.e., $g=bv+c_1$ with b,c_1 are real number. Then by (1.3), constant b satisfies $b^2<1$. The equation (1.6) becomes

$$\left(1-b^2\right)f_{uu}-1+\left(f_u+ab\right)^2+b^2=0.$$

Or

$$\frac{f_{uu}}{1-b^2-\left(f_u+ab\right)^2}=\frac{1}{1-b^2}. \tag{1.11}$$

From the conditional (1.3), it implies that $1-b^2>(f_u+ab)^2>0$. Then, A direct integration yields (1.11)

$$\frac{1}{\sqrt{1-b^2}}\operatorname{arctanh}\frac{f_u+ab}{\sqrt{1-b^2}}=\frac{u}{1-b^2}+d, d\in\mathbb{R}.$$

Therefore,

$$f_u=\sqrt{1-b^2}\tanh\left(\frac{u}{\sqrt{1-b^2}}+c_2\right)-ab, c_2\in\mathbb{R}. \tag{1.12}$$

Furthermore, we have

$$f=(1-b^2)\ln\cosh\left(\frac{u}{\sqrt{1-b^2}}+c_2\right)-abu+c_3, c_3\in\mathbb{R}. \tag{1.13}$$

Checking the conditional (1.3).

From (1.12), we have

$$\left(f_u+ab\right)^2<1-b^2. \tag{1.14}$$

Because $b=g_v$ so (1.14) is equivalent to (1.3).

Substituting $u=x$ and $v=y+ax$ into $f,g,$ the parameter of (S) is (1.4).

+ *Assume that* g_v *is not a constant.*

From (1.10), we have

$$\begin{cases} -\left(1+a^2-f_u^2\right)f_{uuu}+2f_uf_{uu}\left(1+a^2-f_{uu}\right)=0,\\ f_{uu}\left(1+a^2+f_u^2\right)=0,\\ \left(1+a^2-f_u^2\right)f_{uuu}+2f_uf_{uu}\left(a^2+f_{uu}\right)=0. \end{cases}$$

The system of equations above is equivalent to $f_{uu}=0$. Therefore, $f=bu+c_1, b, c_1\in\mathbb{R}$. From (1.3), the constant b must satisfy $b^2<1$. Then, the equation (1.7) becomes

$$\left(1+a^2-b^2\right)g_{vv}-1+\left(b+ag_v\right)^2+g_v^2=0. \tag{1.15}$$

The equation (1.15) is equivalent

$$\frac{g_{vv}}{1-b^2-2abg_v-\left(1+a^2\right)g_v^2}=\frac{1}{1+a^2-b^2}. \tag{1.16}$$

Or

$$\frac{g_{vv}}{\frac{1+a^2-b^2}{1+a^2}-\left(\sqrt{1+a^2}\,g_v+\frac{ab}{\sqrt{1+a^2}}\right)^2}=\frac{1}{1+a^2-b^2}. \tag{1.17}$$

Otherwise, when $f_u=b$, the conditional (1.3) is equivalent to

$1-\left(f'+ag'\right)^2-g'^2>0$ hay $1-b^2-2abg_v-\left(1+a^2\right)g_v^2>0.$

It implies that

$$g_v\in\left(\frac{-ab-\sqrt{1+a^2-b^2}}{1+a^2},\frac{-ab+\sqrt{1+a^2b^2}}{1+a^2}\right). \tag{1.18}$$

A direct integration (1.18) yields

$$\operatorname{arctanh}\frac{\left(a^2+1\right)g_v+ab}{\sqrt{1+a^2-b^2}}=\frac{v}{\sqrt{1+a^2-b^2}}+c_2, c_2\in\mathbb{R}.$$

Or

$$g_v=\frac{\sqrt{1+a^2-b^2}}{a^2+1}\tanh\left(\frac{v}{\sqrt{1+a^2-b^2}}+c_2\right)-\frac{ab}{a^2+1}. \tag{1.19}$$

Continuously, integrating (1.19) yields

$$g=\frac{1+a^2-b^2}{a^2+1}\ln\cosh\left(\frac{v}{\sqrt{1+a^2-b^2}}+c_2\right)-\frac{abv}{a^2+1}+c_3, c_3\in\mathbb{R}. \tag{1.20}$$

From (1.19), the function g_v satisfies (1.18). Therefore, the conditional (1.3) is satisfied.

Substituting $u = x$ and $v = y + ax$ into f, g, reducing, the surface (S) has parameter (1.5).

The Theorem 2.2.2 is proved.

2.3. The image of φ-maximal spacelike affine translation surfaces in the Minkovski space $\mathbb{R}^3_1$ with density e^z

In the Minkowski space $\mathbb{R}^3_1$ with density e^z, Because $\dfrac{d\varphi}{d\nu} = \dfrac{-1}{\sqrt{1-(f' + ag')^2 - g'^2}}$, φ-mean curvature of (S) is invariant under translations by vector $(0,0,d), d \in \mathbb{R}$. Therefore, we can choose $d = 0$ to illustrate images of (S).

2.3.1. The parameter surfaces $X(x,y) = \left(x, y, (1-b^2)\ln\cosh\left(\dfrac{x}{\sqrt{1-b^2}} + c\right) + by\right)$

Two variables x, y of the illustration images are in $(-6,6)\times(-4,4)$. In a Euclidean way, surfaces (S) go to a plane when b^2 goes to 1 or c goes to ∞.

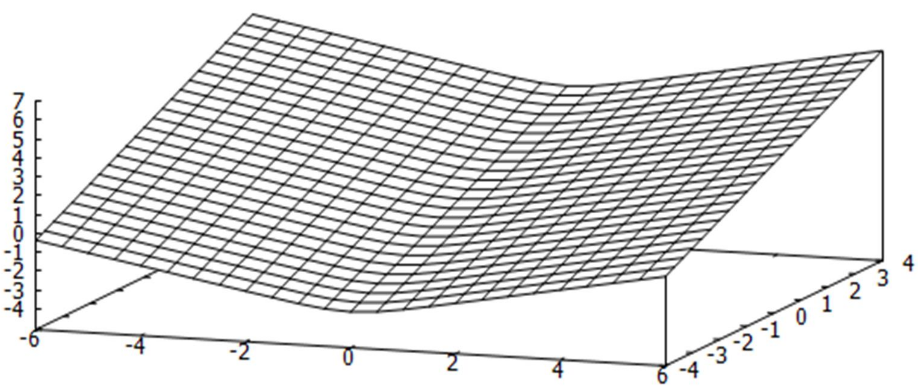

Figure 2.1. $b = 0.8, c = 0.1$

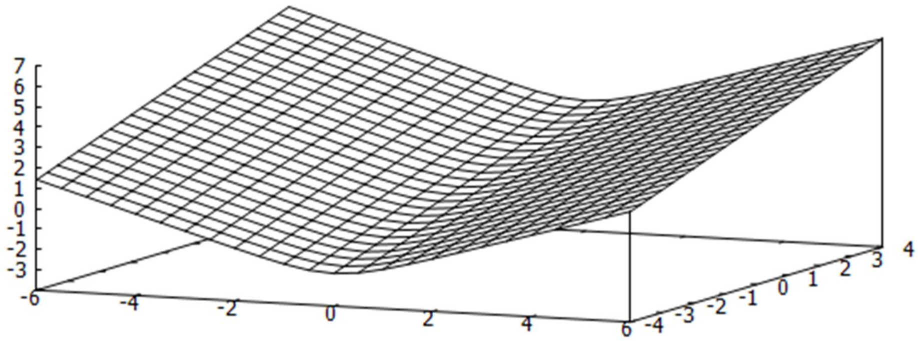

Figure 2.2. $b = 0.6, c = 0.1$

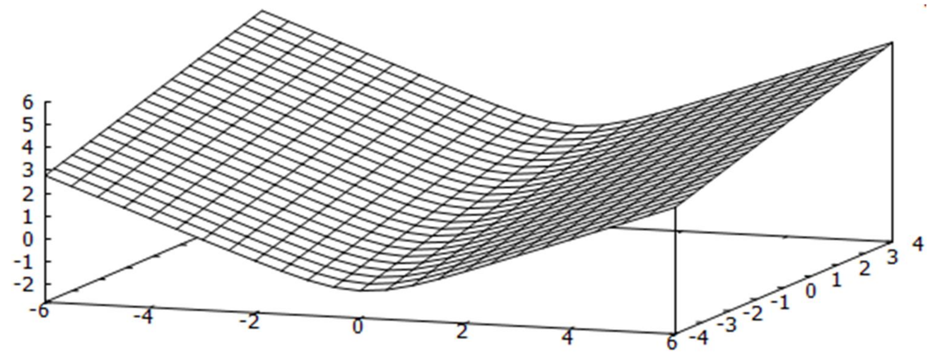

Figure 2.3. $b = 0.4, c = 0.1$

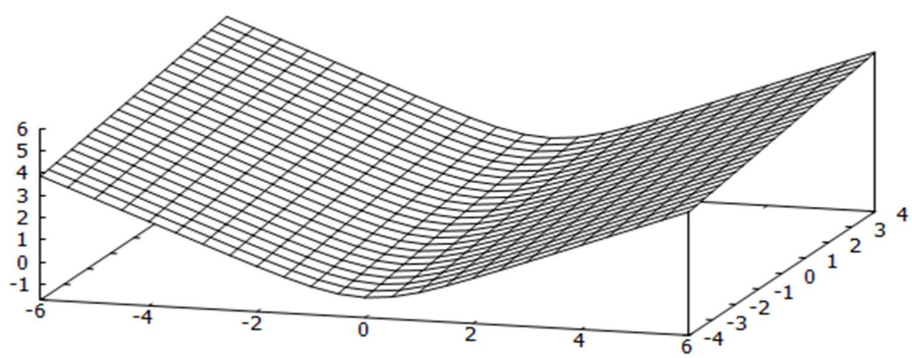

Figure 2.4. $b = 0.2, c = 0.1$

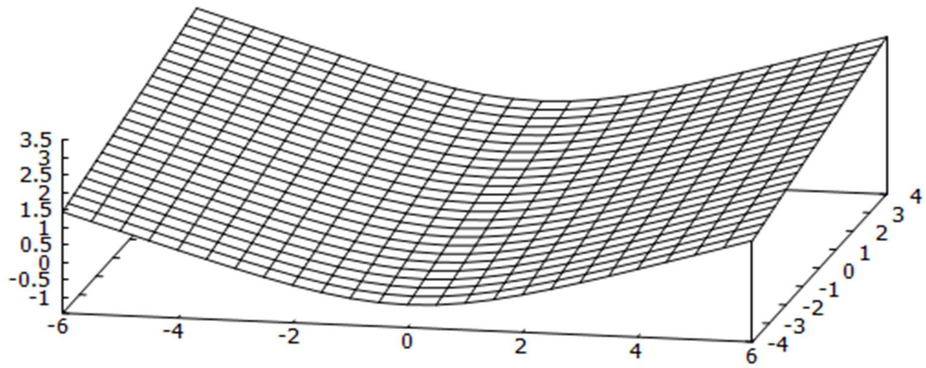

Figure 2.5. $b = 0.2, c = 1.0$

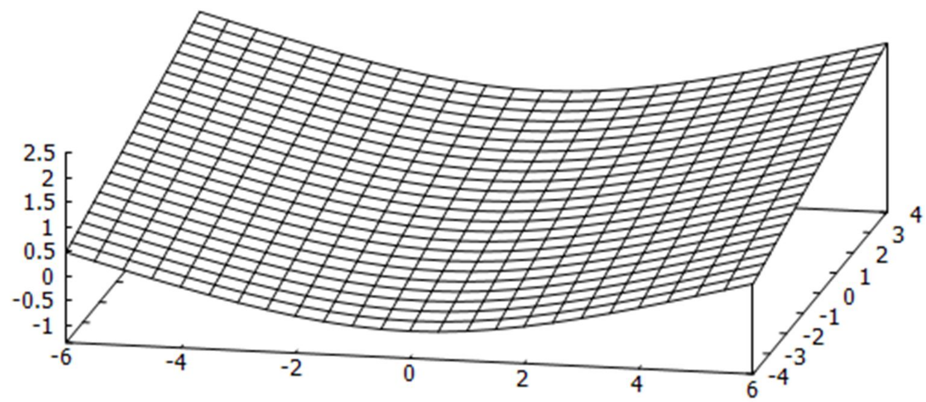

Figure 2.6. $b = 0.2, c = 2.0$

2.3.2. The surfaces $(S): X(x,y) = \left(x, y, \frac{1+a^2-b^2}{a^2+1}\ln\cosh\left(\frac{ax+y}{\sqrt{1+a^2-b^2}} + c\right) + \frac{x-ay}{a^2+1}\right)$

Two variables x,y of the illustration images are in $(-6,6)\times(-4,4)$. In a Euclidean way, surfaces (S) go to a plane when a goes to ∞ or b^2 goes to 1 or c goes to ∞.

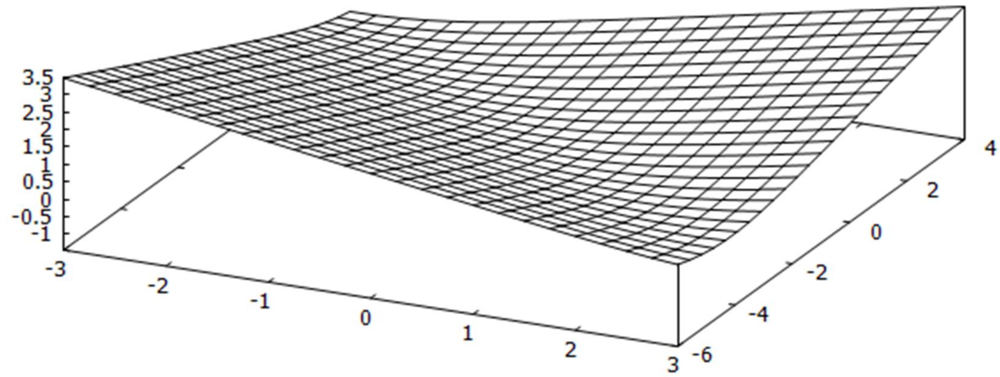

Figure 2.7. $a = 1.5, b = 0.2, c = 0.1$

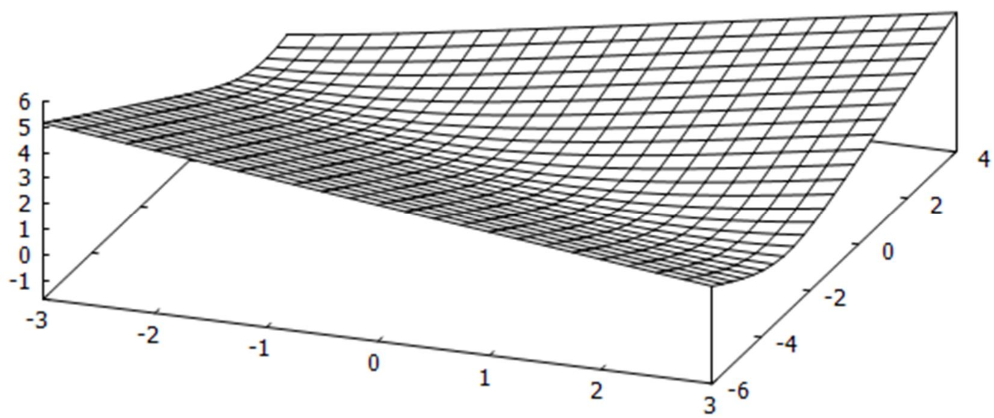

Figure 2.8. $a = 1.0, b = 0.2, c = 0.1$

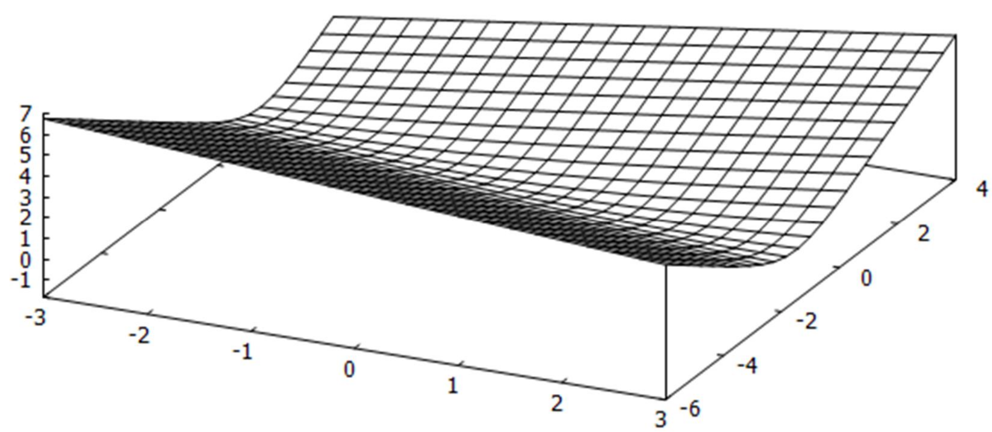

Figure 2.9. $a = 0.5, b = 0.2, c = 0.1$

3. Conclusion

The article classified all j -maximal spacelike affine translation surfaces in the Minkovski space ¡ $^{3}_{1}$ with e^{z}. It shows that their parameters must be (1.4) or (1.5). The classification gives an interesting consequences that the Calabi – Bernstein theorem in this space is not true because these parameters define entire $\mathbf{R}^2$. After that, we draw some illustration images of surface (S). In a Euclidean way, surfaces (S) go to a plane when a goes to ¥ or b^2 goes to 1 or c goes to ¥ . ư

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