THE STRUCTURE OF CONNES' C* – ALGEBRAS ASSOCIATED TO A SUBCLASS OF MD_5 – GROUPS

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ABSTRACT

The paper is a continuation of the authors' works [18], [19]. In [18], we consider foliations formed by the maximal dimensional K-orbits (MD_5-foliations) of connected MD_5-groups that their Lie algebras have 4-dimensional commutative derived ideals and give a topological classification of the considered foliations. In [19], we study K-theory of the leaf space of some of these MD_5-foliations, analytically describe and characterize the Connes' C-algebras of the considered foliations by the method of K-functors. In this paper, we consider the similar problem for all remains of these MD_5-foliations.*

Key words: Lie group, Lie algebra, MD_5-group, MD_5-algebra, K-orbit, Foliation, Measured foliation, C*-algebra, Connes' C*-algebras associated to a measured foliation.

TÓM TẮT

Cấu trúc các C* – đại số Connes liên kết với một lớp con các MD_5 – nhóm

Bài báo này là công trình tiếp nối hai bài báo [18], [19] của các tác giả. Trong [18], chúng tôi đã xét các phân lá tạo thành bởi các K – quỹ đạo chiều cực đại (các MD_5 – phân lá) của các MD_5 – nhóm liên thông mà các đại số Lie của chúng có ideal dẫn xuất giao hoán 4 chiều và đưa ra một phân loại tô pô tất cả các MD_5 – phân lá được xét. Trong [19], chúng tôi đã nghiên cứu K – lý thuyết đối với không gian lá của một vài MD_5 – phân lá trong số đó, mô tả giải tích đồng thời đặc trưng các C – đại số của Connes liên kết với một số phân lá đó bằng phương pháp K – hàm tử. Trong bài này, chúng tôi xét bài toán tương tự đối với tất cả các MD_5 – phân lá còn lại.*

Từ khóa: Nhóm Lie, Đại số Lie, MD5-nhóm, MD5-đại số, K-quỹ đạo, Phân lá, Phân lá đo được, C*-đại số, C*-đại số Connes liên kết với một phân lá đo được.

1. Introduction

In the years of 1970s-1980s, the works of Diep [4], Rosenberg [10], Kasparov [7], Son and Viet [12], … showed that K-functors are well adapted to characterize a large class of group C*-algebras. In 1982, studying foliated manifolds, Connes [3] introduced the notion of C*-algebra associated to a measured foliation. Once again, the method of K-functors has been proved as very effective in describing the structure of Connes' C*-algebras in the case of Reeb foliations (see Torpe [14]).

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Kirillov's method of orbits (see [8, Section 15]) allows to find out the class of Lie groups MD, for which the group C*-algebras can be characterized by means of suitable K- functors (see [5]). Moreover, for every MD-group G, the family of K- orbits of maximal dimension forms a measured foliation in terms of Connes (see [3, Section 2, 5]). This foliation is called MD-foliation associated to G. Recall that an MD-group of dimension n (for short, an MD_n-group), in terms of Diep, is an n-dimensional solvable real Lie group whose orbits in the co-adjoining representation (i.e., the K-representation) are the orbits of zero or maximal dimension. The Lie algebra of an MD_n-group is called an MD_n-algebra (see [5, Section 4.1]).

Combining methods of Kirillov and Connes, the first author studied MD_4-foliations associated with all indecomposable connected MD_4-groups in [16]. Recently, Vu and Shum [17] have classified, up to isomorphism, all the 5-dimensional MD-algebras having commutative derived ideals.

In [18], we have given a topological classification of MD_5-foliations associated to the indecomposable connected and simply connected MD_5-groups, such that MD_5-algebras of them have 4-dimensional commutative derived ideals. There are exactly 3 topological types of considered MD_5-foliations which are denoted by $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. All MD_5-foliations of type $\mathcal{F}_1$ are the trivial fibrations with connected fibre on 3-dimensional sphere S^3, so Connes' C*-algebras $C^*(\mathcal{F}_1)$ of them are isomorphic to the C*-algebra $C\left(S^3\right)\otimes K$ following [3, Section 5], where K denotes the C*-algebra of compact operators on an (infinite dimensional separable) Hilbert space.

In [19], we study K-theory of the leaf space and to characterize the structure of Connes' C*-algebra $C^*(\mathcal{F}_2)$ of all MD_5-foliations of type $\mathcal{F}_2$ by method of K-functors. The purpose of this paper is to study the similar problem for all MD_5-foliations of type $\mathcal{F}_3$. Namely, we will express $C^*(\mathcal{F}_3)$ for all MD_5-foliations of type $\mathcal{F}_3$ by a single extension of the form

$$0 \to C_0\left(X\right)\otimes K \to C^*\left(\mathcal{F}_3\right) \to C_0\left(Y\right)\otimes K \to 0,$$

then we will compute the invariant system of $C^*(\mathcal{F}_3)$ with respect to this extension. Note that if the given C*-algebra is isomorphic to the reduced crossed product of the form $C_0\left(V\right)\rtimes H$, where H is a Lie group, then we can use the Thom-Connes isomorphism to compute the connecting map δ_0, δ_1.

2. The MD_5-foliations of type $\mathcal{F}_3$

Originally, we recall geometry of K-orbits of MD_5-groups which associate with MD_5-foliations of type $\mathcal{F}_3$ (see [17]).

In this section, G will be always one of connected and simply connected MD_5-groups $G_{5,4,14(\lambda,\mu,\varphi)}$ which are studied in [17] and [18]. Then, the Lie algebra G of G will be the one of the Lie algebras $\mathcal{G}_{5,4,14(\lambda,\mu,\varphi)}$ (see [17] or [18]). Namely, G is the

Lie algebra generated by $\{X_1, X_2, X_3, X_4, X_5\}$ with $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \cong \mathbb{R}^4$ and $ad_{X_1} \in End\left(\mathcal{G}^1\right) \equiv Mat_4(\mathbb{R})$ as follows

$$ad_{X_1} := \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{bmatrix}; \quad \lambda, \mu \in \mathbb{R}, \mu > 0, \varphi \in (0, \pi).$$

We now recall the geometric description of the K-orbits of G in the dual space $\mathcal{G}$* of G. Let $\{X_1^*, X_2^*, X_3^*, X_4^*, X_5^*\}$ be the basis in $\mathcal{G}$* dual to the basis $\{X_1, X_2, X_3, X_4, X_5\}$ in G. Denote by Ω_F the K-orbit of G including $F = (\alpha, \beta + i\gamma, \delta + i\sigma)$ in $\mathcal{G}^* \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^5$.

- If $\beta + i\gamma = \delta + i\sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimension orbit),
- If $|\beta + i\gamma|^2 + |\delta + i\sigma|^2 \neq 0$ then Ω_F is the 2-dimension orbit as follows

$$\Omega_F = \left\{ \left(x, (\beta + i\gamma).e^{\left(a.e^{-i\varphi}\right)}, (\delta + i\sigma).e^{a(\lambda - i\mu)} \right),\ x, a \in \mathbb{R} \right\}.$$

In [18], we show that, the family **$\mathcal{F}$** of maximal-dimension K-orbits of G forms measure foliation in terms of Connes on the open sub-manifold

$$V = \left\{ (x, y, z, t, s) \in \mathcal{G}^* : y^2 + z^2 + t^2 + s^2 \neq 0 \right\} \cong \mathbb{R} \times \left(\mathbb{R}^4\right)^* .$$

Furthermore, all the foliations $\left\{ \left(V, \mathcal{F}_{5,4,14(\lambda,\mu,\varphi)}\right),\ \lambda, \mu \in \mathbb{R}, \mu > 0, \varphi \in (0; \pi) \right\}$, are topologically equivalent to each other and we denote them by **$\mathcal{F}_3$** . So we only choose a “envoy” among them to describe the structure of *C*(F*$_3$*)* by K-functors. In this case, we choose the foliation $\left(V, \mathcal{F}_{5,4,14\left(0,1,\frac{\pi}{2}\right)}\right)$.

In [18], we also describe the foliation $\left(V, \mathcal{F}_{5,4,14\left(0,1,\frac{\pi}{2}\right)}\right)$ by suitable action of $\mathbb{R}^2$. Namely, we have the following assertion.

Proposition 2.1. *The foliation* $\left(V, \mathcal{F}_{5,4,14\left(0,1,\frac{\pi}{2}\right)}\right)$ *can be given by an action of the commutative Lie group* $\mathbb{R}^2$ *on the manifold V.*

Proof. One needs only to verify that the foliation $\left(V, \mathcal{F}_{5,4,14\left(0,1,\frac{\pi}{2}\right)}\right)$ is given by the action $\lambda : \mathbb{R}^2 \times V \to V$ of $\mathbb{R}^2$ on V as follows

$$\lambda\left(\left(r, a\right), \left(x, y + iz, t + is\right)\right) := \left(x + r, (y + iz).e^{-ia}, (t + is).e^{-ia}\right),$$

where $(r, a) \in \mathbb{R}^2$ and $(x, y + iz, t + is) \in V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{C})^* \cong \mathbb{R} \times (\mathbb{R}^4)^*$. Hereafter, for simply, we write $\mathbf{F}_3$ instead of $\left(V, \mathcal{F}_{5,4,14\left(0,1,\frac{\pi}{2}\right)}\right)$.

It is easy to see that the graph of $\mathbf{F}_3$ is identified with $V \times \mathbb{R}^2$, so by [3, Section 5], it follows from Proposition 2.1 that

Corollary 2.2. *(Analytical description of C*($\mathbf{F}_3$)) The Connes' C*-algebra C*($\mathbf{F}_3$) can be analytically described by the reduced crossed product of $C_0(V)$ by $\mathbb{R}^2$ as follows*

$$C^*(\mathbf{F}_3) \cong C_0(V) \rtimes_\lambda \mathbb{R}^2.$$

3. C*($\mathbf{F}_3$) as a single extension

3.1. Let V_1, W_1 be the following sub-manifolds of V

$$V_1 = \left\{(x, y + iz, t + is) \in V : t + is \neq 0\right\} \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}^*,$$

$$W_1 = V \setminus V_1 = \left\{(x, y + iz, t + is) \in V : t + is = 0\right\} \cong \mathbb{R} \times \mathbb{C}^*.$$

It is easy to see that the action λ in Proposition 2.1 preserves the subsets V_1, W_1. Let i, μ be the inclusion and the restriction

$$i : C_0(V_1) \to C_0(V), \qquad \mu : C_0(V) \to C_0(W_1).$$

where each function of $C_0(V_1)$ is extended to the one of $C_0(V)$ by taking the value of zero outside V_1.

It is known a fact that i, μ are λ- equivariant and the following sequence is equivariantly exact:

$$(3.1) \qquad 0 \longrightarrow C_0(V_1) \xrightarrow{\ i\ } C_0(V) \xrightarrow{\ \mu\ } C_0(W_1) \longrightarrow 0.$$

3.2. Now we denote by $(V_1, \mathcal{F}_1)$, $(W_1, \mathcal{F}_1)$ ***restrictions of the foliations*** $\mathbf{F}_3$ ***on*** V_1, W_1***, respectively.***

Theorem 3.1. *C*($\mathbf{F}_3$) admits the following canonical extension*

$$(\gamma_1) \qquad 0 \longrightarrow J \xrightarrow{\ \hat{i}\ } C^*(\mathbf{F}_3) \xrightarrow{\ \hat{\mu}\ } B \longrightarrow 0,$$

where $J = C^*(V_1, \mathcal{F}_1) \cong C_0(V_1) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{C} \times \mathbb{R}_+) \otimes K$,

$B = C^*(W_1, \mathcal{F}_1) \cong C_0(W_1) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{R}_+) \otimes K$,

$C^*(\mathcal{F}_3) \cong C_0(V) \rtimes_\lambda \mathbb{R}^2$.

and the homomorphism $\hat{i}$, $\hat{\mu}$ *is defined by*

$$\left(\hat{i} f\right)(r,s) = if(r,s), \quad \left(\hat{\mu} f\right)(r,s) = \mu f(r,s).$$

Proof. Note that the graph of $\mathcal{F}_3$ is identified with $V \times \mathbb{R}^2$, so by [3, section 5], we have:

$J = C^*(V_1, \mathcal{F}_1) \cong C_0(V_1) \rtimes_\lambda \mathbb{R}^2$,

$B = C^*(W_1, \mathcal{F}_1) \cong C_0(W_1) \rtimes_\lambda \mathbb{R}^2$.

From λ-equivariantly exact sequence in 3.1 and by [2, Lemma 1.1] we obtain the single extension (γ_1). Furthermore, the foliations $(V_1, \mathcal{F}_1)$ and $(W_1, \mathcal{F}_1)$ can be come from the submersions

$$p : V \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{R}_+ \qquad \text{and} \qquad q : W \cong \mathbb{R} \times \mathbb{C}^* \to \mathbb{R}_+$$

$$\left(x, re^{i\varphi}, r'e^{i\varphi'}\right) \mapsto \left(re^{i\varphi}, r'\right) \qquad \left(x, re^{i\varphi}\right) \mapsto r$$

Hence, by a result of [3, p.562], we get

$J = C^*(V_1, \mathcal{F}_1) \cong C_0(V_1) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{C} \times \mathbb{R}_+) \otimes K$,

$B = C^*(W_1, \mathcal{F}_1) \cong C_0(W_1) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{R}_+) \otimes K$.

4. Computing the invariant system of $C^*(\mathcal{F}_3)$

Definition 4.1. The set of element $\{\gamma_1\}$ corresponding to the single extension (γ_1) in the Kasparov group $\mathrm{Ext}(B, J)$ is called *the system of invariant* of $C^*(\mathcal{F}_3)$ and denoted by Index $C^*(\mathcal{F}_3)$.

Remark 4.2. Index $C^*(\mathcal{F}_3)$ determines the so-called table type of $C^*(\mathcal{F}_3)$ in the set of all single extension

$$0 \longrightarrow J \longrightarrow E \longrightarrow B \longrightarrow 0.$$

The main result of the paper is the following

Theorem 4.3. *Index* $C^*(\mathcal{F}_3) = \{\gamma_1\}$, *where*

$\gamma_1 = (0,1)$ *in the group* $Ext(B, J) = Hom(\mathbb{Z}, \mathbb{Z}) \oplus Hom(\mathbb{Z}, \mathbb{Z})$.

To prove this theorem, we need some lemmas as follows

Lemma 4.4. *Set* $I = C_0\left(\mathbb{R}^2 \times S^1\right)$ and $A = C\left(S^1\right)$.

The following diagram is commutative

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & K_j(I) & \longrightarrow & K_j\left(C\left(S^3\right)\right) & \longrightarrow & K_j(A) & \longrightarrow & K_{j+1}(I) & \longrightarrow \ldots \\
 & \downarrow \beta_2 & & \downarrow \beta_2 & & \downarrow \beta_2 & & \downarrow \beta_2 & \\
\ldots \longrightarrow & K_j\left(C_0\left(V_1\right)\right) & \longrightarrow & K_j\left(C_0(V)\right) & \longrightarrow & K_j\left(C_0\left(W_1\right)\right) & \longrightarrow & K_{j+1}\left(C_0\left(V_1\right)\right) & \longrightarrow \ldots
\end{array}$$

where β_2 *is the Bott isomorphism,* $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. Let

$$k : C_0\left(\mathbb{R}^2 \times S^1\right) \longrightarrow C\left(S^3\right), \qquad v : C\left(S^3\right) \longrightarrow C\left(S^1\right).$$

be the inclusion and restriction defined similarly as in 3.1.

One gets the exact sequence

$$0 \longrightarrow I \xrightarrow{k} C\left(S^3\right) \xrightarrow{v} A \longrightarrow 0.$$

Note that

$$C_0\left(V_1\right) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+\right) \otimes C_0\left(\mathbb{R}^2 \times S^1\right) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+\right) \otimes I$$

$$C_0(V) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+ \times S^3\right) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+\right) \otimes C\left(S^3\right)$$

$$C_0\left(W_1\right) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+\right) \otimes C\left(S^1\right) \cong C_0\left(\mathbb{R} \times \mathbb{R}_+\right) \otimes A$$

So, the extension (3.1) can be identified to the following one

$$0 \to C_0\left(\mathbb{R}_+ \times \mathbb{R}\right) \otimes I \xrightarrow{Id \otimes k} C_0\left(\mathbb{R}_+ \times \mathbb{R}\right) \otimes C\left(S^3\right) \xrightarrow{Id \otimes v} C_0\left(\mathbb{R}_+ \times \mathbb{R}\right) \otimes A \to 0.$$

So, the assertion of lemma is derived from the naturalness of Bott isomorphism.

Remark 4.5.

i) $K_j\left(C_0\left(\mathbb{R}^2 \times S^1\right)\right) \cong K_j\left(C_0\left(S^1\right)\right) \cong \mathbb{Z}$, $j \in \mathbb{Z}/2\mathbb{Z}$.

ii) $K_j\left(C\left(S^3\right)\right) \cong \mathbb{Z}$, $j \in \mathbb{Z}/2\mathbb{Z}$.

iii) $K_0\left(C\left(S^1\right)\right) \cong \mathbb{Z}$ *is generated by* $\varphi_0 \beta_2 [1]$, $K_1\left(C\left(S^1\right)\right) \cong \mathbb{Z}$ *is generated by* $\varphi_1 \beta_2 [Id]$ *(where 1 is a unit element in* $C\left(S^1\right)$*;* φ_j, $j \in \mathbb{Z}/2\mathbb{Z}$ *, is the Thom-Connes isomorphism; Id is the identity of* S^1*).*

Proof of Theorem 4.3. Recall that the extension (γ_1) in theorem 3.1 gives the rise to a six-term exact sequence

$$(4.1)\quad \begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(C^*(\mathbf{F}_3)) & \longrightarrow & K_0(B) \\ \delta_1\uparrow & & & & \downarrow\delta_0 \\ K_1(B) & \longleftarrow & K_1(C^*(\mathbf{F}_3)) & \longleftarrow & K_1(J) \end{array}$$

By [11, Theorem 4.14], the isomorphism

$$Ext(B,J) \cong Hom_{\mathbb{Z}}(K_0(B),K_1(J)) \oplus Hom_{\mathbb{Z}}(K_1(B),K_0(J))$$

associates the invariant $\gamma_1 \in Ext(B,J)$ to the pair

$$(\delta_0,\delta_1) \in Hom_{\mathbb{Z}}(K_0(B),K_1(J)) \oplus Hom_{\mathbb{Z}}(K_1(B),K_0(J)).$$

Since the Thom-Connes isomorphism commutes with K-theoretical exact sequence (see [14, Lemma 3.4.3]), we have the following commutative diagram $(j \in \mathbb{Z}/2\mathbb{Z})$:

$$\begin{array}{ccccccccccc} \ldots\longrightarrow & K_j(J) & \longrightarrow & K_j(C^*(\mathbf{F}_3)) & \longrightarrow & K_j(B) & \longrightarrow & K_{j+1}(J) & \longrightarrow\ldots \\ & \uparrow\varphi_j & & \uparrow\varphi_j & & \uparrow\varphi_j & & \uparrow\varphi_{j+1} & \\ \ldots\longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_j(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) & \longrightarrow & K_{j+1}(C_0(V_1)) & \longrightarrow\ldots \end{array}$$

In view of Lemma 4.4, the following diagram is commutative

$$\begin{array}{ccccccccccc} \ldots\longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_j(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) & \longrightarrow & K_{j+1}(C_0(V_1)) & \longrightarrow\ldots \\ & \uparrow\beta_2 & & \uparrow\beta_2 & & \uparrow\beta_2 & & \uparrow\beta_2 & \\ \ldots\longrightarrow & K_j(I) & \longrightarrow & K_j(C(S^3)) & \longrightarrow & K_j(A) & \longrightarrow & K_{j+1}(I) & \longrightarrow\ldots \end{array}$$

Consequently, instead of computing the pair (δ_0,δ_1) from the direct sum $Hom_{\mathbb{Z}}(K_0(B),K_1(J)) \oplus Hom_{\mathbb{Z}}(K_1(B),K_0(J))$, it is sufficient to compute the pair $(\delta_0,\delta_1) \in Hom_{\mathbb{Z}}(K_0(A),K_1(I)) \oplus Hom_{\mathbb{Z}}(K_1(A),K_0(I))$. In other words, the six-term exact sequence (4.1) can be identified with the following one

$$(4.2)\quad \begin{array}{ccccc} K_0(C_0(\mathbb{R}^2 \times S^1)) & \longrightarrow & K_0(C(S^3)) & \longrightarrow & K_0(C(S^1)) \\ \delta_1\uparrow & & & & \downarrow\delta_0 \\ K_1(C(S^1)) & \longleftarrow & K_1(C(S^3)) & \longleftarrow & K_1(C_0(\mathbb{R}^2 \times S^1)) \end{array}$$

By remark 3.5, this sequence becomes

$$(4.3)\quad \begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \delta_1\uparrow & & & & \downarrow\delta_0 \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \end{array}$$

By the exactness, the sequence (4.3) will be the one of the following ones

$$\begin{array}{ccccc} \square & \xrightarrow{0} & \square & \xrightarrow{1} & \square \\ \delta_1 = 1 \uparrow & & & & \downarrow \delta_0 = 0 \\ \square & \xleftarrow{0} & \square & \xleftarrow{1} & \square \end{array}$$

or

$$\begin{array}{ccccc} \square & \xrightarrow{1} & \square & \xrightarrow{0} & \square \\ \delta_1 = 0 \uparrow & & & & \downarrow \delta_0 = 1 \\ \square & \xleftarrow{1} & \square & \xleftarrow{0} & \square \end{array}$$

Now we choose $a = e^{i\varphi} \in GL_1\left(C\left(S^1\right)\right)$, $b = a^{-1}$. Then

$$a \oplus b = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix} \in GL_2^0\left(C\left(S^1\right)\right).$$

Let u = $u\left(x, y, z, t\right) = u\left(\cos\theta_1 \cos\theta_2 \cos\varphi, \cos\theta_1 \cos\theta_2 \sin\varphi, \cos\theta_1 \sin\theta_2, \sin\theta_1\right)$

$$= \begin{bmatrix} e^{i\varphi}.e^{i\theta_1}.\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & e^{-i\varphi}.e^{-i\theta_1}.\cos\theta_2 \end{bmatrix} \in GL_2^0\left(C\left(S^3\right)\right)$$

is a pre-image of $a \oplus b$. So, $u^{-1} = \begin{bmatrix} e^{-i\varphi} e^{-i\theta_1} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & e^{i\varphi} e^{i\theta_1} \cos\theta_2 \end{bmatrix}$.

Let $q = I_1 \oplus 0_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We get

$$p = uqu^{-1} = \begin{bmatrix} \cos^2\theta_2 & e^{i\varphi} e^{i\theta_1} \cos\theta_2 \sin\theta_2 \\ e^{-i\varphi} e^{-i\theta_1} \cos\theta_2 \sin\theta_2 & \sin^2\theta_2 \end{bmatrix} \in P_2\left(C_0\left(\square^2 \times S^1\right)\right)^+.$$

Then *rank (p)* = 1. So $\delta_1\left(\left[a\right]\right) = \left[p\right] - \left[I_1\right] \neq 0 \in K_0\left(C_0\left(\square^2 \times S^1\right)\right)$. Therefore, K-theoretical exact sequence associate to $\left(\gamma_1\right)$ is

$$\begin{array}{ccccc} \square & \xrightarrow{0} & \square & \xrightarrow{1} & \square \\ \delta_1 = 1 \uparrow & & & & \downarrow \delta_0 = 0 \\ \square & \xleftarrow{0} & \square & \xleftarrow{1} & \square \end{array}$$

The proof is completed.

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