SOME EXTENSIONS FROM A QUADRATIC LIE ALGEBRA

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ABSTRACT

In this paper, we give some extensions from a quadratic Lie algebra to quadratic Lie superalgebras and odd-quadratic Lie superalgebras. Moreover, we use the cohomology to recover some results obtained by the method of double extension.

Keywords: Quadratic Lie algebras, Quadratic Lie superalgebras, Odd-quadratic Lie superalgebras, Extensions, Symplectic structure.

TÓM TẮT

Một số mở rộng từ đại số Lie toàn phương

Trong bài báo này, chúng tôi sẽ đưa ra một số mở rộng từ một đại số Lie toàn phương lên siêu đại số Lie toàn phương và siêu đại số Lie toàn phương lẻ. Bên cạnh đó chúng tôi sử dụng công cụ đối đồng điều để chứng minh lại số kết quả thu được từ phương pháp mở rộng kép.

Từ khóa: đại số Lie toàn phương, siêu đại số Lie toàn phương, siêu đại số Lie toàn phương lẻ, mở rộng, cấu trúc symplectic.

1. Introduction

Let g be a complex Lie algebra and g^* its dual space. Denote by ad and ad^* the adjoint and coadjoint representations of g, respectively. It is known that the semidirect product $\overline{g} = g \oplus g^*$ of g and g^* by ad^* is a Lie algebra with the bracket given by:

$$\left[X + f, Y + g\right] = \left[X, Y\right] + ad^*(X)(g) - ad^*(Y)(f), \ \forall X, Y \in g, \ f, g \in g^*.$$

More particularly, we have $\left[X, Y\right]_{\overline{g}} = \left[X, Y\right]_g$, $\left[X, f\right]_{\overline{g}} = -f \circ ad(X)$ and $\left[f, g\right]_{\overline{g}} = 0$ for all $X, Y \in g$, $f, g \in g^*$. Remark that $\overline{g}$ is also a quadratic Lie algebra with invariant symmetric bilinear form B defined by:

$$B(X + f, Y + g) = f(Y) + g(X), \ \forall X, Y \in g, \ f, g \in g^*.$$

In 1985, A. Medina and P. Revoy gave the notion of double extension to completely characterize all quadratic Lie algebras [9]. This notion is regarded as a generalization of the definition of semidirect product by the coadjoint representation. Another generalization is called T*-extension given by M. Bordemann that is suffictent

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to describe solvable quadratic Lie algebras [4]. We will recall them and some basic results in Section 2. Sections 3 and 4 are devoted to give an expansion of these two notions for Lie superalgebras. In particular, we present a way to obtain a quadratic Lie superalgebra since a Lie algebra and a symplectic vector space. It is regarded as a rather special case of the notion of generalized double extension in [1]. In a slight change of the notion of T*-extension, we give a manner of how to get an odd-quadratic Lie superalgebra from a Lie algebra.

In the last section, we introduce an approach to quadratic Lie algebras by the cohomology given in [9] and [10]. From this, we give an explanation of the structure of double extension as well as it allows us to construct new quadratic Lie algebra structures from a given quadratic Lie algebra.

2. Quadratic Lie algebras

Definintion 2.1. Let g be a Lie algebra. A bilinear form $B: g \times g \to £$ is called:

- *(i)* *symmetric* if $B(X,Y) = B(Y,X)$ for all $X, Y \in g$,
- *(ii)* *non-degenerate* if $B(X,Y) = 0$ for all $Y \in g$ implies $X = 0$,
- *(iii)* *invariant* if $B([X,Y],Z) = B(X,[Y,Z])$ for all $X, Y, Z \in g$.

A Lie algebra g is called *quadratic* if there exists a bilinear form B on g such that B is symmetric, non-degenerate and invariant.

Definition 2.2. Let (g, B) be a quadratic Lie algebra and D be a derivation of g. We say D a *skew-symmetric* derivation of g if it satisfies

$$B(D(X),Y) = -B(X,D(Y)),\ \forall X, Y \in g.$$

Denote by $Der_a(g,B)$ the vector space of skew-symmetric derivations of (g,B) then $Der_a(g,B)$ is a subalgebra of $Der(g)$, the Lie algebra of derivations of g. The notion of double extension is defined as follows (see also in [9]).

Definition 2.3. Let g be a Lie algebra, g^* be its dual space and (h,B) be a quadratic Lie algebra. Let $\phi : g \to Der_a(h,B)$ be a Lie algebra endomorphism. Denote by $\varphi : h \times h \to g^*$ the linear mapping defined by:

$$\varphi(X,Y)Z = B(\phi(Z)(X),Y),\ \forall X,Y \in h,\ Z \in g.$$

Consider the vector space $\bar{h} = g \oplus h \oplus g^*$ and define a product on $\bar{h}$:

$$\left[X + F + f, Y + G + g\right]_{\bar{h}} = \left[X,Y\right]_g + \left[F,G\right]_h + ad^*(X)(g) - ad^*(Y)(f) + \phi(X)(G) - \phi(Y)(F) + \varphi(F,G)$$

for all $X,Y \in \mathfrak{g}$, $f,g \in \mathfrak{g}^*$ and $F,G \in \mathfrak{h}$. Then $\bar{\mathfrak{h}}$ becomes a quadratic Lie algebra with the bilinear form $\bar{B}$ given by:

$$\bar{B}(X+F+f, Y+G+g) = f(Y) + g(X) + B(F,G)$$

for all $X,Y \in \mathfrak{g}$, $f,g \in \mathfrak{g}^*$ and $F,G \in \mathfrak{h}$. The Lie algebra $(\bar{\mathfrak{h}}, \bar{B})$ is called the *double extension of* $(\mathfrak{h}, B)$ *by* $\mathfrak{g}$ *by means of* ϕ.

Note that when $\mathfrak{h} = \{0\}$ then this definition is reduced to the notion of the semidirect product of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the coadjoint representation.

Proposition 2.4. ([8], 2.11, [9], Theorem I). *Let* $\mathfrak{g}$ *be an indecomposable quadratic Lie algebra such that it is not simple nor one-dimensional. Then* $\mathfrak{g}$ *is the double extension of a quadratic Lie algebra by a simple or one-dimensional algebra.*

Sometimes, we use a particular case of the notion of double extension, that is a double extension by a skew-symmetric derivation. It is explicitly defined as follows.

Definition 2.5. Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and $C \in \mathrm{Der}_a(\mathfrak{g}, B)$. On the vector space $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e \oplus \mathbb{C}f$ we define the product:

$$[X,Y]_{\bar{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} + B(C(X),Y)f,\ [e,X] = C(X) \text{ and } [f, \bar{\mathfrak{g}}] = 0$$

for all $X,Y \in \mathfrak{g}$. Then $\bar{\mathfrak{g}}$ is a quadratic Lie algebra with invariant bilinear form $\bar{B}$ defined by:

$$\bar{B}(e,e) = \bar{B}(f,f) = \bar{B}(e,\mathfrak{g}) = \bar{B}(f,\mathfrak{g}) = 0,\ \bar{B}(X,Y) = B(X,Y) \text{ and } B(e,f) = 1$$

for all $X,Y \in \mathfrak{g}$. In this case, we call $\bar{\mathfrak{g}}$ the double extension of $\mathfrak{g}$ by C or a one-dimensional double extension, for short.

The one-dimensional double extensions are sufficient for studying solvable quadratic Lie algebras by the following proposition (see [6] or [8]).

Proposition 2.6. *Let* $(\mathfrak{g}, B)$ *be a solvable quadratic Lie algebra of dimension* n, $n \geq 2$. *Assume* $\mathfrak{g}$ *non-Abelian. Then* $\mathfrak{g}$ *is a one-dimensional double extension of a solvable quadratic Lie algebra of dimension* $n-2$.

We give now another generalization given by M. Bordemann as follows.

Definition 2.7. Let $\mathfrak{g}$ be a Lie algebra and $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ be a 2-cocycle of $\mathfrak{g}$, that is a skew-symmetric bilinear map satisfying:

$$\theta(X,Y) \circ \mathrm{ad}(Z) + \theta([X,Y],Z) + \mathrm{cycle}(X,Y,Z) = 0$$

for all $X,Y,Z \in \mathfrak{g}$. Define on the vector space $T_\theta^*(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^*$ the following product:

$$\left[X+f, Y+g\right]=\left[X, Y\right]+\mathrm{ad}^*(X)(g)-\mathrm{ad}^*(Y)(f)+\theta(X, Y)$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$. Then $T_\theta^*(\mathfrak{g})$ becomes a Lie algebra and it is called the T*-extension of $\mathfrak{g}$ by means of θ. In addition, if θ satisfies the cyclic condition, i.e. $\theta(X, Y)Z=\theta(Y, Z)X$ for all $X, Y, Z \in \mathfrak{g}$ then $T_\theta^*(\mathfrak{g})$ is quadratic with the bilinear form:

$$B(X+f, Y+g)=f(Y)+g(X), \ \forall X, Y \in \mathfrak{g}, \ f, g \in \mathfrak{g}^*.$$

Note that in the case of $\theta=0$ then this notion is exactly the semidirect product by the coadjoint representation.

Proposition 2.8. [4]. *Let* $(\mathfrak{g}, B)$ *be an even-dimensional quadratic Lie algebra over* £ . *If* $\mathfrak{g}$ *is solvable then it is i-isomorphic to a T*-extension* $T_\theta^*(\mathfrak{h})$ *of* $\mathfrak{h}$ *where* $\mathfrak{h}$ *is the quotient algebra of* $\mathfrak{g}$ *by a totally isotropic ideal*.

3. Quadratic Lie superalgebras

Definition 3.1. Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. If there is a non-degenerate supersymmetric bilinear form B on $\mathfrak{g}$ such that B is even and invariant then the pair $(\mathfrak{g}, B)$ is called a *quadratic Lie superalgebra*.

Note that if $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra then $\mathfrak{g}_{\bar{0}}$ is a quadratic Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a symplectic vector space with the restriction of the bilinear form B on each part.

Lemma 3.2. *Let* $\mathfrak{g}$ *be a Lie algebra and* $(\mathfrak{h}, B_{\mathfrak{h}})$ *a symplectic vector space with symplectic form* $B_{\mathfrak{h}}$. *Let* $\psi: \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ *be a Lie algebra endomorphism satisfying:*

$$B_{\mathfrak{h}}(\psi(X)(Y), Z)=-B_{\mathfrak{h}}(Y, \psi(X)(Z)), \ \forall X \in \mathfrak{g}, \ Y, Z \in \mathfrak{h}.$$

Denote by $\phi: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}^*$ *the bilinear map defined by:*

$$\phi(X, Y)Z=B_{\mathfrak{h}}(\psi(Z)(X), Y), \ \forall X, Y \in \mathfrak{h}, \ Z \in \mathfrak{g}.$$

Then ϕ *is symmetric, i.e.* $\phi(X, Y)=\phi(Y, X)$ *for all* $X, Y \in \mathfrak{h}$.

Proof. For all $X, Y \in \mathfrak{h}$, $Z \in \mathfrak{g}$,

$$\phi(X, Y)Z=B_{\mathfrak{h}}(\psi(Z)(X), Y)=-B_{\mathfrak{h}}(X, \psi(Z)(Y))=B_{\mathfrak{h}}(\psi(Z)(Y), X)=\phi(Y, X)Z.$$

Then one has $\phi(X, Y)=\phi(Y, X)$.

□

Theorem 3.3. *Keep notions as in the above lemma and define on the vector space* $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{h}$ *the following bracket:*

$$\left[X + f + F, Y + g + G\right]_{\overline{\mathfrak{g}}} = \left[X, Y\right]_{\mathfrak{g}} + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) + \psi(X)(G) - \psi(Y)(F) + \phi(F, G)$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$. *Then* $\overline{\mathfrak{g}}$ *becomes a quadratic Lie superalgebra with* $\overline{\mathfrak{g}}_{\bar{0}} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\overline{\mathfrak{g}}_{\bar{1}} = \mathfrak{h}$ *and the bilinear form* $\overline{B}$ *defined by:*

$$\overline{B}(X + f + F, Y + g + G) = f(Y) + g(X) + B_{\mathfrak{h}}(F, G)$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$.

Proof. We check first that the bracket satisfies the super-antisymmetric property $\left[Y, X\right] = -(-1)^{xy}[Y, X]$, $\forall X \in \overline{\mathfrak{g}}_x, Y \in \overline{\mathfrak{g}}_y$. Indeed, if $X + f, Y + g \in \overline{\mathfrak{g}}_{\bar{0}}$ then

$$\left[X + f, Y + g\right] = \left[X, Y\right]_{\mathfrak{g}} + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) = -\left[Y + g, X + f\right].$$

If $X + f \in \overline{\mathfrak{g}}_{\bar{0}}, Y \in \overline{\mathfrak{g}}_{\bar{1}}$ then $\left[X + f, Y\right] = \psi(X)(Y)$ and $\left[Y, X + f\right] = -\psi(X)(Y)$.

And if $X \in \overline{\mathfrak{g}}_{\bar{1}}, Y \in \overline{\mathfrak{g}}_{\bar{1}}$ then $\left[X, Y\right] = \phi(X, Y) = \phi(Y, X) = [X, Y]$. Therefore, one has $\left[Y, X\right] = -(-1)^{xy}[Y, X]$, $\forall X \in \overline{\mathfrak{g}}_x, Y \in \overline{\mathfrak{g}}_y$.

Next, we check the Jacobi identity:

$$(-1)^{zx}\left[X, \left[Y, Z\right]\right] + (-1)^{xy}\left[Y, \left[Z, X\right]\right] + (-1)^{yz}\left[Z, \left[X, Y\right]\right] = 0$$

is right for all $X \in \overline{\mathfrak{g}}_x$, $Y \in \overline{\mathfrak{g}}_y$ and $Z \in \overline{\mathfrak{g}}_z$. Indeed, if $X, Y, Z \in \overline{\mathfrak{g}}_{\bar{0}}$ then the Jacobi identity is clear. Let $X + f, Y + g \in \overline{\mathfrak{g}}_{\bar{0}}$ and $Z \in \overline{\mathfrak{g}}_{\bar{1}}$ then

$$\left[X + f, \left[Y + g, Z\right]\right] = \left[X + f, \psi(Y)(Z)\right] = \psi\left(X\right)\left(\psi(Y)(Z)\right) = \psi\left(X\right) \mathrm{o} \psi(Y)(Z),$$

$$\left[Y + g, \left[Z, X + f\right]\right] = \left[Y + g, -\psi(X)(Z)\right] = -\psi\left(Y\right)\left(\psi(X)(Z)\right) = -\psi\left(Y\right) \mathrm{o} \psi(X)(Z)$$

and

$$\left[Z, \left[X + f, Y + g\right]\right] = \left[Z, \left[X, Y\right] + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f)\right] = -\psi\left(\left[X, Y\right]\right)(Z).$$

Therefore, $\left[X + f, \left[Y + g, Z\right]\right] + \left[Y + g, \left[Z, X + f\right]\right] + \left[Z, \left[X + f, Y + g\right]\right] = 0$.

If $X + f \in \overline{\mathfrak{g}}_{\bar{0}}$ and $Y, Z \in \overline{\mathfrak{g}}_{\bar{1}}$ then

$$\left[X + f, \left[Y, Z\right]\right] = \left[X + f, \phi(Y, Z)\right] = \mathrm{ad}^*(X)\left(\phi(Y, Z)\right) = -\phi(Y, Z) \, \mathrm{o} \, \mathrm{ad}(X),$$

$$\left[Y, \left[Z, X + f\right]\right] = \left[Y, -\psi(X)(Z)\right] = -\phi(Y, \psi(X)(Z))$$

and $-\left[Z,\left[X+f,Y\right]\right]=-\left[Z,\psi(X)(Y)\right]=-\phi(Z,\psi(X)(Y))$.

To prove $-\phi(Y,Z)\circ\mathrm{ad}(X)-\phi(Y,\psi(X)(Z))-\phi(Z,\psi(X)(Y))=0$, let $T\in\mathfrak{g}$ we have:

$$-\phi(Y,Z)\circ\mathrm{ad}(X)(T)=-\phi(Y,Z)\left(\left[X,T\right]\right)=-B_{\mathfrak{h}}\left(\psi\left(\left[X,T\right]\right)(Y),Z\right),$$

$$-\phi(Y,\psi(X)(Z))(T)=-B_{\mathfrak{h}}\left(\psi(T)(Y),\psi(X)(Z)\right)=B_{\mathfrak{h}}\left(\psi(X)\circ\psi(T)(Y),Z\right) \text{ and}$$

$$-\phi(Z,\psi(X)(Y))(T)=-B_{\mathfrak{h}}\left(\psi(T)(Z),\psi(X)(Y)\right)=B_{\mathfrak{h}}\left(\psi(X)(Y),\psi(T)(Z)\right)$$
$$=-B_{\mathfrak{h}}\left(\psi(T)\circ\psi(X)(Y),Z\right).$$

Therefore $-\phi(Y,Z)\circ\mathrm{ad}(X)-\phi(Y,\psi(X)(Z))-\phi(Z,\psi(X)(Y))=0$ and then

$$\left[X+f,\left[Y,Z\right]\right]+\left[Y,\left[Z,X+f\right]\right]-\left[Z,\left[X+f,Y\right]\right]=0.$$

If $X,Y,Z\in\overline{\mathfrak{g}}_{\bar{1}}$ then the Jacobi identity is obviously satisfied since

$$\left[X,\left[Y,Z\right]\right]=\left[X,\phi(Y,Z)\right]=0.$$

In final, we shall check $\overline{B}$ invariant. This is a straightforward computation. It is easy to see that $\overline{B}$ is symmetric on $\overline{\mathfrak{g}}_{\bar{0}}\times\overline{\mathfrak{g}}_{\bar{0}}$, skew-symmetric on $\overline{\mathfrak{g}}_{\bar{1}}\times\overline{\mathfrak{g}}_{\bar{1}}$ and vanish on $\overline{\mathfrak{g}}_{\bar{0}}\times\overline{\mathfrak{g}}_{\bar{1}}$. Hence, we can conclude that $\overline{\mathfrak{g}}$ is a quadratic Lie superalgebra.

◻

Now we combine Definition 2.7 and Theorem 3.3 to get a more general result as follows.

Theorem 3.4. *Let $\mathfrak{g}$ be a Lie algebra and $\theta:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}^*$ a 2-cocycle of $\mathfrak{g}$. Assume $\left(\mathfrak{h},B_{\mathfrak{h}}\right)$ a symplectic vector space with symplectic form $B_{\mathfrak{h}}$. Let $\psi:\mathfrak{g}\to\mathrm{End}(\mathfrak{h})$ be a Lie algebra endomorphism satisfying:*

$$B_{\mathfrak{h}}\left(\psi(X)(Y),Z\right)=-B_{\mathfrak{h}}\left(Y,\psi(X)(Z)\right),\ \forall X\in\mathfrak{g},\ Y,Z\in\mathfrak{h}$$

Denote by $\phi:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{g}^*$ the bilinear map defined by:

$$\phi(X,Y)Z=B_{\mathfrak{h}}\left(\psi(Z)(X),Y\right),\ \forall X,Y\in\mathfrak{h},\ Z\in\mathfrak{g}$$

and define on the vector space $\overline{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{g}^*\oplus\mathfrak{h}$ the following bracket:

$$\left[X+f+F,Y+g+G\right]_{\overline{\mathfrak{g}}}=\left[X,Y\right]_{\mathfrak{g}}+\mathrm{ad}^*(X)(g)-\mathrm{ad}^*(Y)(f)+\theta(X,Y)$$
$$+\psi(X)(G)-\psi(Y)(F)+\phi(F,G)$$

for all $X,Y \in \mathfrak{g}$, $f,g \in \mathfrak{g}^*$ *and* $F,G \in \mathfrak{h}$. *Then* $\overline{\mathfrak{g}}$ *becomes a quadratic Lie superalgebra with* $\overline{\mathfrak{g}}_{\bar{0}} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\overline{\mathfrak{g}}_{\bar{1}} = \mathfrak{h}$.

As a consequence of Definition 2.7 and Theorem 3.4, we also have the following corollary.

Corollary 3.5. *If* θ *is cyclic then* $\overline{\mathfrak{g}}$ *is a quadratic Lie superalgebra with the bilinear form:*

$$\overline{B}\left(X+F+f, Y+G+g\right) = f(Y) + g(X) + B_{\mathfrak{h}}(F,G)$$

for all $X,Y \in \mathfrak{g}$, $f,g \in \mathfrak{g}^*$ *and* $F,G \in \mathfrak{h}$.

4. Odd-quadratic Lie superalgebras

Definition 4.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. If there is a non-degenerate supersymmetric bilinear form B on $\mathfrak{g}$ such that B is odd and invariant then the pair $(\mathfrak{g}, B)$ is called an *odd-quadratic Lie superalgebra*.

Let $\mathfrak{g}$ be a Lie algebra and $\varphi : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}$ be a bilinear map. We define on the vector space $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ the following bracket:

$$\left[X+f, Y+g\right] = \left[X,Y\right] + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) + \varphi(f,g)$$

for all $X,Y \in \mathfrak{g}$, $f,g \in \mathfrak{g}^*$. We search some condition such that $\overline{\mathfrak{g}}$ becomes a Lie super algebra with $\overline{\mathfrak{g}}_{\bar{0}} = \mathfrak{g}$ and $\overline{\mathfrak{g}}_{\bar{1}} = \mathfrak{g}^*$. In this case, φ is obviously symmetric.

(i) $\left[X,\left[f,g\right]\right] = \left[X,\varphi\left(f,g\right)\right]$, $\left[f,\left[g,X\right]\right] = \varphi\left(f, g \circ \mathrm{ad}(X)\right)$ and $-\left[g,\left[X,f\right]\right] = \varphi\left(g, f \circ \mathrm{ad}(X)\right)$. Therefore, one must have

$$\mathrm{ad}(X)\left(\varphi\left(f,g\right)\right) + \varphi\left(f, g \circ \mathrm{ad}(X)\right) + \varphi\left(g, f \circ \mathrm{ad}(X)\right) = 0.$$

(ii) $\left[f,\left[g,h\right]\right] = f \circ \mathrm{ad}\left(\varphi\left(g,h\right)\right)$, $\left[h,\left[f,g\right]\right] = h \circ \mathrm{ad}\left(\varphi\left(f,g\right)\right)$ and $\left[g,\left[h,f\right]\right] = g \circ \mathrm{ad}\left(\varphi\left(h,f\right)\right)$.

Then we hace the second condition:

$$f \circ \mathrm{ad}\left(\varphi\left(g,h\right)\right) + g \circ \mathrm{ad}\left(\varphi\left(h,f\right)\right) + h \circ \mathrm{ad}\left(\varphi\left(f,g\right)\right) = 0.$$

For the bilinear form $\overline{B}$ is defined by:

$$\overline{B}(X+f, Y+g) = f(Y) + g(X), \ \forall X,Y \in \mathfrak{g}, \ f,g \in \mathfrak{g}^*,$$

one has:

$$\overline{B}\left(\left[X+f,Y+g\right]_{\overline{\mathfrak{g}}},Z+h\right)=h\left(\left[X,Y\right]+\varphi(f,g)\right)+g\left(\left[Z,X\right]\right)-f\left(\left[Z,Y\right]\right)$$

and $B\left(X+f,\left[Y+g,Z+h\right]_{\overline{\mathfrak{g}}}\right)=f\left(\left[Y,Z\right]+\varphi(g,h)\right)+h\left(\left[X,Y\right]\right)-g\left(\left[X,Z\right]\right)$. Hence it must have $h(\varphi(f,g))=f(\varphi(g,h))$.

Finally, we have the following result.

Theorem 4.2. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\varphi:\mathfrak{g}^*\times\mathfrak{g}^*\to\mathfrak{g}$ *a symmetric bilinear map satisfying two conditions:*

(i) $\mathrm{ad}(X)\left(\varphi(f,g)\right)+\varphi\left(f,g\circ\mathrm{ad}(X)\right)+\varphi\left(g,f\circ\mathrm{ad}(X)\right)=0,$

(ii) $f\circ\mathrm{ad}\left(\varphi(g,h)\right)+g\circ\mathrm{ad}\left(\varphi(h,f)\right)+h\circ\mathrm{ad}\left(\varphi(f,g)\right)=0$

for all $X,Y\in\mathfrak{g},\ f,g\in\mathfrak{g}^*$.

Then the vector space $\overline{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{g}^*$ *with the following bracket:*

$$\left[X+f,Y+g\right]=\left[X,Y\right]+\mathrm{ad}^*(X)(g)-\mathrm{ad}^*(Y)(f)+\varphi(f,g)$$

for all $X,Y\in\mathfrak{g},\ f,g\in\mathfrak{g}^*$ *is a Lie superalgebra and called the* T_s^**-extension of* $\mathfrak{g}$ *by means of* φ*. Moreover, if* φ *satisfies the condition* $h(\varphi(f,g))=f(\varphi(g,h))$ *then* $\overline{\mathfrak{g}}$ *is an odd-quadratic with the bilinear form*

$$B(X+f,Y+g)=f(Y)+g(X),\ \forall X,Y\in\mathfrak{g},\ f,g\in\mathfrak{g}^*.$$

5. Approach to quadratic Lie algebras by the structure equation

5.1. The associatied 3-form and the structure equation

Given a finite dimensional complex vector space V, equipped with a non-degenerate symmetric bilinear form B. In [10], G. Pinczon and R. Ushirobira introduced the notion of the super Poisson bracket on the exterior algebra $\Lambda\left(V^*\right)$ as follows $\{\Omega,\Omega'\}=(-1)^{k+1}\sum_{j=1}^{n}\iota_{X_j}(\Omega)\wedge\iota_{X_j}(\Omega'),\quad\forall\Omega\in\Lambda^k\left(V^*\right)$ và $\Omega'\in\Lambda\left(V^*\right)$ with $\left\{X_j\right\}_{j=1}^{n}$ a fixed orthonormal basis of V. For a quadratic Lie algebra $(\mathfrak{g},B)$, they defined a trilinear form I by $I(X,Y,Z)=B([X,Y],Z)$ for all $X,Y,Z\in\mathfrak{g}$ then $\{I,I\}=0$. Moreover, the quadratic Lie algebra structure of $(\mathfrak{g},B)$ is completely characterized by I and there is a one-to-one correspondence between the set of structures of quadratic Lie algebra and the set of I satisfying $\{I,I\}=0$. Then we call I the *associated 3-form* and $\{I,I\}=0$ the *structure equation* of $(\mathfrak{g},B)$.

Recall that $Der_a(g,B)$ of skew-symmetric derivations of g is a Lie subalgebra of $Der(g)$ of derivations of g.

Proposition 5.1. [9]. *There exists a natural isomorphism* T *between* $Der_a(g,B)$ *and the space* $\left\{\Omega \in \Lambda^2\left(g^*\right) : \{I,\Omega\}=0\right\}$ *that induces an isomorphism from* $Der_a(g,B)/ad(g)$ *onto the second cohomology group* $H^2(g,\mathbb{C})$.

Next, we shall use this isomorphism to construct a new structure of quadratic Lie algebra from g as follows. Let I_g be the associated 3-form of g and assume $\Omega \in \Lambda^2\left(g^*\right)$ such that $\{I_g,\Omega\}=0$. On the vector space $\overline{g} = g \oplus \mathbb{C}e \oplus \mathbb{C}f$ we extend B to $\overline{B}$ such that $\overline{B}(e,f)=1$ and $\overline{B}(e,e)=\overline{B}(f,f)=0$. Set $\alpha=\overline{B}(f,.)$ and define $I_{\overline{g}} = \alpha \wedge \Omega + I_g$.

Theorem 5.2.

(i) *The element* $I_{\overline{g}}$ *defines a quadratic Lie algebra structure on* $\overline{g}$.

(ii) *The element* $I_{\overline{g}}$ *is the associated 3-form of the double extension of* $\overline{g}$ *by* $T^{-1}(\Omega)$.

Proof.

(i) One has $\left\{I_{\overline{g}}, I_{\overline{g}}\right\} = \left\{\alpha\wedge\Omega, \alpha\wedge\Omega\right\} + 2\left(\left\{I_g,\alpha\right\}\wedge\Omega - \alpha\wedge\left\{I_g,\Omega\right\}\right) + \left\{I_g, I_g\right\} = 0$.

(ii) For all $X, Y \in g$, by [10], $[X,Y]_{\overline{g}} = \iota_{X\wedge Y}\left(I_{\overline{g}}\right)$ then $[e,X]\in g$ and $[X,Y]_{\overline{g}} \in g \oplus \mathbb{C}f$.

Also, $I_{\overline{g}}(X,Y,Z) = I(X,Y,Z)$ so $\overline{B}\left([X,Y]_{\overline{g}},Z\right) = B\left([X,Y]_g,Z\right)$ and then

$[X,Y]_{\overline{g}} = [X,Y]_g + \Omega(X,Y)f$.

Let $C = T^{-1}(\Omega)$ then

$\overline{B}\left(e,[X,Y]\right) = I_{\overline{g}}(e,X,Y) = \alpha\left(e\right)\Omega(X,Y) = B(C(X),Y)$.

It means $\Omega(X,Y) = B(C(X),Y)$. By the invariance of B, one has $[e,X]=C(X)$. So that $I_{\overline{g}}$ defines the double extension of g by C.

Remark 5.3.

(i) In the case g Abelian, i.e. $I_g = 0$ then it is obviously $\{I_g,\Omega\}=0$ for any 2-form Ω on g and therefore $I_{\overline{g}} = \alpha\wedge\Omega$. This case has been studied in [5].

(ii) If $C = \mathrm{ad}(X)$ is an inner derivation of $\mathfrak{g}$ then the double extension $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ by C has $I_{\bar{\mathfrak{g}}} = \alpha \wedge \iota_X(I) + I$. In this case, $\iota_{e-X}\left(I_{\bar{\mathfrak{g}}}\right) = 0$. It means $e - X$ central and then we recover a result in [7] that $\bar{\mathfrak{g}}$ is decomposable.

5.2. Symplectic quadratic Lie algebras

Definition 5.4. Given a Lie algebra $\mathfrak{g}$. A non-degenerate skew-symmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called a *symplectic structure* on $\mathfrak{g}$ if it satisfies

$$\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0,\ \forall X,Y,Z \in \mathfrak{g}.$$

A symplectic structure ω on a quadratic Lie algebra $(\mathfrak{g}, B)$ is corresponding to a skew-symmetric invertible derivation D defined by $\omega(X,Y) = B(D(X),Y)$ for all $X, Y \in \mathfrak{g}$. As above, a symplectic structure is exactly a non-degenerate 2-form ω satisfying $\{I, \omega\} = 0$. In this case, we call $(\mathfrak{g}, B, \omega)$ a symplectic quadratic Lie algebra.

Let $(\mathfrak{g}, B, \omega)$ be a symplectic quadratic Lie algebra. Assume Ω is a non-degenerate 2-form satisfying $\{I, \Omega\} = 0$ and Ω' is another 2-form satisfying $\{I, \Omega'\} = 0$. The double extension $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ corresponding to Ω' has $I_{\bar{\mathfrak{g}}} = \alpha \wedge \Omega' + I$. We set a non-degenerate 2-form on $\bar{\mathfrak{g}}$ by $\Omega_{\bar{\mathfrak{g}}} = \Omega + \lambda e^* \wedge \left(f^* + X^*\right)$ with $\lambda \neq 0$ and some $X \in \mathfrak{g}$. We search a condition of Ω, Ω', λ and X such that $\Omega_{\bar{\mathfrak{g}}}$ define a skew-symmetric invertible derivation on $\bar{\mathfrak{g}}$ and therefore it defines a symplectic structure on $\bar{\mathfrak{g}}$. By the condition $\{I, \Omega\} = 0$ one has

$$0 = \left\{I_{\bar{\mathfrak{g}}}, \Omega_{\bar{\mathfrak{g}}}\right\} = \left\{e^* \wedge \Omega' + I, \Omega + \lambda e^* \wedge \left(f^* + X^*\right)\right\}$$
$$= \left\{e^* \wedge \Omega', \Omega\right\} + \left\{e^* \wedge \Omega', \lambda e^* \wedge \left(f^* + X^*\right)\right\} + \left\{I, \lambda e^* \wedge \left(f^* + X^*\right)\right\}$$

Since $\left\{e^* \wedge \Omega', \Omega\right\} = -e^* \wedge \left\{\Omega, \Omega'\right\}$, $\left\{e^* \wedge \Omega', \lambda e^* \wedge \left(f^* + X^*\right)\right\} = -\lambda e^* \wedge \Omega'$ and $\left\{I, \lambda e^* \wedge \left(f^* + X^*\right)\right\} = -\lambda e^* \wedge \iota_X(I)$ so we have $\left\{\Omega, \Omega'\right\} + \lambda \Omega' + \lambda \iota_X(I) = 0$. Set $D = T^{-1}(\Omega)$, $\delta = T^{-1}(\Omega')$ and $X_0 = \lambda X$ so we obtain

$$[D, \delta] + \lambda\delta + \mathrm{ad}\left(X_0\right) = 0$$

and then we recover Lemma 4.1 in [3].

Note that in the above case, if we choose $\Omega_{\bar{\mathfrak{g}}} = \Omega + \lambda f^* \wedge \left(e^* + X^*\right)$ then by a similar computation we have $\Omega' = 0$ and X_0 central. So the above condition is obvious.

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