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IDEAL CO-TRANSFORMS AND LOCAL HOMOLOGY WITH RESPECT TO A PAIR OF IDEALS

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ABSTRACT

We introduce the concept of the ideal co-transforms with respect to a pair of ideal (I, J) *which is an extension of the concept of the ideal co-transforms* [1] *and dual to the concept of the ideal transform of Brodmann* [2]. *We also study some basic properties of the ideal co-transforms with respect to a pair of ideal* (I, J) *for linearly compact modules*.

Keywords: ideal transforms, linearly compact module, local homology.

TÓM TẮT

Đối biến đổi iđêan và Đồng điều địa phương cho một cặp iđêan

Chúng tôi sẽ giới thiệu khái niệm đối biến đổi iđêan cho một cặp iđêan (I, J), *xem như là một sự mở rộng từ khái niệm đối biến đổi iđêan trong bài báo* [1] *và đối ngẫu với khái niệm biến đổi iđêan của Brodmann* [2]. *Đồng thời, chúng tôi sẽ nghiên cứu một số tính chất cơ bản của đối biến đổi iđêan cho trường hợp môđun compact tuyến tính.*

Từ khóa: biến đổi iđêan, môđun compact tuyến tính, đồng điều địa phương.

1. Introduction

Throughout this paper, R is a noetherian commutative ring and has a topological structure. Let I, J be two ideals of R. In [2], Brodmann introduced the definition of ideal transform $D_I(M)$ of an $R-$module M with respect to ideal I

$$D_I(M) = \varinjlim_t \operatorname{Hom}_R(I^t, M).$$

It provides a powerful tool in commutative algebra. Some extensions of $D_I(M)$ are the generalized ideal transforms of M, N with respect to an ideal $D_I(M, N)$ in [3] and ideal transforms with respect to a pair of ideals $D_{I,J}(M)$ in [4].

By the duality, in [1] Tran Tuan Nam defined the concept of ideal co-transforms $C_i^I(M)$ of an $R-$module M by

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$$C_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(I^t, M),$$

the author also studied some properties of ideal co-transforms $C_i^I(M)$ in case M is a linearly compact $R-$module. In [5], we studied the local homology modules with respect to a pair of ideal (I, J) defined by

$$H_i^{I,J}(M) = \varprojlim_{\mathfrak{a}} H_i^{\mathfrak{a}}(M).$$

This is an extension of local cohomology modules and dual to local cohomology modules with respect to a pair of ideal (I, J) [6]. By duality, we define ideal co-transforms with respect to a pair of ideals $C_i^{I,J}(M)$ by

$$C_i^{I,J}(M) = \varprojlim_{\mathfrak{a}} C_i^{\mathfrak{a}}(M).$$

The purpose of this paper is to study the ideal co-transforms with respect to a pair of ideal $C_i^{I,J}(M)$ in the case M is a linearly compact $R-$module. The organization of the paper is as follows. In section 2, we recall briefly some properties of linearly compact modules and local homology modules that we shall use. In section 3, we introduce the concept of the ideal co-transforms with respect to a pair of ideal $C_i^{I,J}(M)$ and show some properties of the ideal co-transforms $C_i^{I,J}(M)$ when M is a linearly compact $R-$module. Theorem 3.3 shows the relationships between $C_i^{I,J}(M)$ and $H_i^{I,J}(M)$ through the isomorphisms $H_{i+1}^{I,J}(M) \cong C_i^{I,J}(M)$ for all $i \geq 1$ and the exact sequence

$$0 \to H_1^{I,J}(M) \to C^{I,J}(M) \to M \to \Lambda_{I,J}(M) \to 0.$$

Theorem 3.7 gives us some basic properties of the ideal co-transforms $C_i^{I,J}(M)$. Finally, Theorem 3.8 shows the property of the ideal co-transforms $C_i^{I,J}(M)$ related to the continuous homomorphisms.

2. Preliminaries

We first recall briefly definitions and basic properties of linearly compact modules and local homology modules that we shall use.

Let M be a topological $R-$module. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [7]). It is clear that artinian $R-$modules are linearly compact and discrete. We have some following properties of linearly compact modules.

Lemma 2.1. ([7])

i) If M is a Hausdorff linearly topologized $R-$module and N a closed submodule of M, then M is linearly compact if and only if N and M/N are linearly compact.

ii) Let $f: M \to N$ be a continuous homomorphism of Hausdorff linearly topologized $R-$modules. If M is linearly compact, then $f(M)$ is linearly compact and f is a closed map.

iii) The inverse limit of a system of linearly compact $R-$modules with continuous homomorphisms is linearly compact.

Lemma 2.2. ([8, 7.1])

Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphism and

$$0 \to \{M_t\} \to \{N_t\} \to \{P_t\} \to 0$$

be a short exact sequence of inverse systems of $R-$modules. Then

$$0 \to \varprojlim_t M_t \to \varprojlim_t N_t \to \varprojlim_t P_t \to 0$$

is exact.

Let I, J be two ideals of the ring R and M an $R-$module. The $i-$th local homology module $H_i^{I,J}(M)$ of M with respect to a pair of ideals (I, J) is defined in [5] by

$$H_i^{I,J}(M) = \varprojlim_{\mathfrak{a}} H_i^{\mathfrak{a}}(M).$$

This definition is in some sense dual to the definition of local cohomology modules $H_{I,J}^i(M)$ of M with respect to a pair of ideals (I, J) ([6]).

The $(I, J)-$adic completion $\Lambda_{I,J}(M)$ of an $R-$module M is defined by

$$\Lambda_{I,J}(M) = \varprojlim_{\mathfrak{a}} \Lambda_{\mathfrak{a}}(M).$$

It is clear that if $J = 0$, then $\Lambda_{I,J}(M) = \Lambda_I(M)$, which is the $I-$adic completion of M.

Lemma 2.3. ([5, 2.9])

Let $(R, \mathfrak{m})$ be a local ring and M an artinian $R-$module. If M is complete with respect to the $J-$adic topology (i.e., $\Lambda_J(M) \cong M$), then

$$H_i^{I,J}(M) \cong H_i^I(M)$$

for all $i \geq 0$.

Lemma 2.4. ([5, 2.13, 2.14])

Let M be a linearly compact $R-$module. Then

i) $H_i^{I,J}(H_j^{I,J}(M)) \cong \begin{cases} H_j^{I,J}(M), & i=0 \\ 0, & i>0 \end{cases}$, for all $j \geq 0$;

ii) $H_i^{I,J}\left(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M\right) \cong \begin{cases} 0, & i=0 \\ H_i^{I,J}(M), & i>0 \end{cases}$.

3. Ideal Co-Transforms

Let I, J be ideals of R, we recall $\tilde{W}(I,J)$ the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a}+J$ for some integer n ([6]). We define a partial order on $\tilde{W}(I,J)$ by $\mathfrak{a} \leq \mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$. The ideal transform of M with respect to ideals (I,J) is defined by $D_{I,J}(M) = \varinjlim_{\mathfrak{a}} D_{\mathfrak{a}}(M)$ ([4]). If $\mathfrak{a} \leq \mathfrak{b}$ we have the homomorphisms $\operatorname{Tor}_i^R(\mathfrak{b}^t, M) \to \operatorname{Tor}_i^R(\mathfrak{a}^t, M)$ for all $t>0$ and $i \geq 0$. It induces a homomorphism $C_i^{\mathfrak{b}}(M) \to C_i^{\mathfrak{a}}(M)$. Therefore, we have an inverse system $\{C_i^{\mathfrak{a}}(M)\}_{\mathfrak{a}\in \tilde{W}(I,J)}$. Now, we define the ideal co-transform with respect to a pair of ideal, which dual to the definition of ideal transform.

Definition 3.1.

Let M be an $R-$module and I, J ideals of R. The $i-$*ideal co-transform* of M with respect to a pair of ideals (I,J) or $(I,J)-$co-transform of M is defined by

$$C_i^{I,J}(M) = \varprojlim_{\mathfrak{a}\in \tilde{W}(I,J)} C_i^{\mathfrak{a}}(M).$$

When $i=0$, $C_0^{I,J}(M)$ is called the $(I,J)-$co-transform of M and denoted by $C^{I,J}(M)$.

If M is linearly compact $R-$module, then by [1], $C_i^{\mathfrak{a}}(M)$ is also linearly compact, and $C_i^{I,J}(M)$ is too by Lemma 2.1, (iii).

Proposition 3.2.

Let $0 \to M'' \xrightarrow{f} M \xrightarrow{g} M' \to 0$ be a short exact sequence of linearly compact $R-$modules in which the homomorphisms f, g are continuous. Then we have a long exact sequence of linearly compact $R-$modules

$$\cdots \longrightarrow C_i^{I,J}(M'') \xrightarrow{f_i} C_i^{I,J}(M) \xrightarrow{g_i} C_i^{I,J}(M') \longrightarrow \cdots$$

$$\longrightarrow C^{I,J}(M'') \xrightarrow{f_0} C^{I,J}(M) \xrightarrow{g_0} C^{I,J}(M') \longrightarrow 0$$

in which the homomorphims f_i, g_i are continuous for all $i \geq 0$.

Proof. By [1, 3.2], we have the exact sequence of linearly compact $R-$modules

$$\cdots \to C_i^{\mathfrak{a}}(M'') \xrightarrow{f_{i_{\mathfrak{a}}}} C_i^{\mathfrak{a}}(M) \xrightarrow{g_{i_{\mathfrak{a}}}} C_i^{\mathfrak{a}}(M') \longrightarrow \cdots$$
$$\to C^{\mathfrak{a}}(M'') \xrightarrow{f_{0\mathfrak{a}}} C^{\mathfrak{a}}(M) \xrightarrow{g_{0\mathfrak{a}}} C^{\mathfrak{a}}(M') \to 0$$

in which $f_{i_{\mathfrak{a}}}, g_{i_{\mathfrak{a}}}$ are continuous for all $\mathfrak{a} \in \widetilde{W}(I,J)$. Then $\operatorname{Im} f_{i_{\mathfrak{a}}}, \operatorname{Im} g_{i_{\mathfrak{a}}}$ are linearly compact. Hence, $\varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)}$ is exact on all of the short exact sequence that arise from the long sequence. Therefore, we have the long exact sequence in the theorem. Since the homomorphisms $f_{i_{\mathfrak{a}}}, g_{i_{\mathfrak{a}}}$ are continuous, the homomorphisms induced on corresponding direct products are also continuous, so f_i, g_i are continuous.

Theorem 3.3.

i) *For all $R-$module M and $i \geq 1$, $H_{i+1}^{I,J}(M) \cong C_i^{I,J}(M)$;*

ii) *If M is linearly compact $R-$module, then there is an exact sequence of linearly compact modules*

$$0 \to H_1^{I,J}(M) \to C^{I,J}(M) \to M \to \Lambda_{I,J}(M) \to 0.$$

Proof. i) We have

$$H_{i+1}^{I,J}(M) = \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H_{i+1}^{\mathfrak{a}}(M)$$
$$C_i^{I,J}(M) = \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} C_i^{\mathfrak{a}}(M).$$

By [1, 3.3], we have $H_{i+1}^{\mathfrak{a}}(M) \cong C_i^{\mathfrak{a}}(M)$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$. Therefore, $H_{i+1}^{I,J}(M) \cong C_i^{I,J}(M)$.

ii) For each $\mathfrak{a} \in \widetilde{W}(I,J)$, we have the exact sequence

$$0 \to H_1^{\mathfrak{a}}(M) \to C^{\mathfrak{a}}(M) \xrightarrow{\eta_{\mathfrak{a}}} M \xrightarrow{\theta_{\mathfrak{a}}} \Lambda_{\mathfrak{a}}(M) \to 0$$

in which $\eta_{\mathfrak{a}}, \theta_{\mathfrak{a}}$ are continuous for all $\mathfrak{a} \in \widetilde{W}(I,J)$ by [1, 3.3(ii)]. From [9, 3.3], $\{H_1^{\mathfrak{a}}(M)\}$ is linearly compact inverse system and so is $\{\operatorname{Ker}\theta_{\mathfrak{a}}\}$. Passing into inverse limits $\varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)}$, we have an exact sequence

$$0 \to H_1^{I,J}(M) \to C^{I,J}(M) \to M \to \Lambda_{I,J}(M) \to 0.$$

Corollary 3.4.

Let M be a linearly compact $R-$module. Then $C^{I,J}(M) \cong M$ if and only if $H_1^{I,J}(M) = \Lambda_{I,J}(M) = 0$.

Proof. It follows from Theorem 3.3 (ii).

Corollary 3.5.

Let M be a linearly compact $R-$module. There are two short exact sequences

$$0 \to H_1^{I,J}(M) \to C^{I,J}(M) \to \bigcap_{\mathfrak{a}\in \widetilde{W}(I,J)} \mathfrak{a}M \to 0$$

$$0 \to \bigcap_{\mathfrak{a}\in \widetilde{W}(I,J)} \mathfrak{a}M \to M \to \Lambda_{I,J}(M) \to 0.$$

Proof. There is a continuous homomorphism $\theta_{\mathfrak{a}} : M \to \Lambda_{\mathfrak{a}}(M)$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$ and $\operatorname{Ker}\theta_{\mathfrak{a}} = \bigcap_{t>0} \mathfrak{a}^t M$ is inverse system of linearly compact modules.

Hence, from the short exact sequence

$$0 \to \bigcap_{t>0} \mathfrak{a}^t M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0$$

we have the exact sequence

$$0 \to \bigcap_{\mathfrak{a}\in \widetilde{W}(I,J)} \mathfrak{a}M \to M \to \Lambda_{I,J}(M) \to 0.$$

From the Theorem 3.3, we have

$$0 \to H_1^{I,J}(M) \to C^{I,J}(M) \to \bigcap_{\mathfrak{a}\in \widetilde{W}(I,J)} \mathfrak{a}M \to 0.$$

Proposition 3.6.

Let $(R,\mathfrak{m})$ be a local ring and M is an artinian $R-$module. If M is $J-$adic completion, then $C^{I,J}(M) \cong C^I(M)$.

Proof. By Lemma 2.3, we have $H_i^{I,J}(M) \cong H_i^I(M)$ for all $i \geq 0$. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & H_1^I(M) & \to & C^I(M) & \to & M & \to & \Lambda_I(M) & \to 0 \\
 & \downarrow\cong & & \downarrow & & \downarrow \text{id} & & \downarrow\cong & \\
0 \to & H_1^{I,J}(M) & \to & C^{I,J}(M) & \to & M & \to & \Lambda_{I,J}(M) & \to 0,
\end{array}$$

where two rows are exact sequences by Theorem 3.3 and [1]. This implies that $C^{I,J}(M) \cong C^I(M)$.

Theorem 3.7.

Let M be a linearly compact $R-$module. Then

i) $C^{I,J}(H_i^{I,J}(M)) = 0$ for all $i \geq 0$;

ii) $C^{I,J}(\bigcap_{\mathfrak{a}\in \widetilde{W}(I,J)} \mathfrak{a}M) \cong C^{I,J}(M)$;

iii) $C^{I,J}(M) \cong C^{I,J}(C^{I,J}(M))$;

iv) $H_i^{I,J}(C^{I,J}(M)) \cong H_i^{I,J}(M)$ for all $i \geq 1$;

v) $\Lambda_{I,J}(C^{I,J}(M)) = H_1^{I,J}(C^{I,J}(M)) = 0.$

Proof. i) As M is linearly compact module, so is $H_i^{I,J}(M)$. From Theorem 3.3 (ii), we have the exact sequence

$$0 \to H_1^{I,J}(H_i^{I,J}(M)) \to C^{I,J}(H_i^{I,J}(M)) \to H_i^{I,J}(M)$$
$$\to \Lambda_{I,J}(H_i^{I,J}(M)) \to 0.$$

By Lemma 2.4, $H_1^{I,J}(H_i^{I,J}(M))=0$ and $\Lambda_{I,J}(H_i^{I,J}(M)) \cong H_i^{I,J}(M)$, hence $C^{I,J}(H_i^{I,J}(M))=0$ for all $i \geq 0$.

ii) From the second exact sequence of Corollary 3.5 and Proposition 3.2, there is an exact sequence

$$C_1^{I,J}(\Lambda_{I,J}(M)) \to C^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) \to C^{I,J}(M) \to C^{I,J}(\Lambda_{I,J}(M)) \to 0.$$

By Theorem 3.3, $C_1^{I,J}(\Lambda_{I,J}(M)) \cong H_2^{I,J}(\Lambda_{I,J}(M))=0$ and $C^{I,J}(\Lambda_{I,J}(M))=0$ by (i). Therefore $C^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) \cong C^{I,J}(M)$.

iii) The first exact sequence of Corollary 3.5 induces the exact

$$C^{I,J}(H_1^{J,J}(M)) \to C^{I,J}(C^{I,J}(M)) \to C^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) \to 0.$$

Since $C^{I,J}(H_1^{J,J}(M))=0$ by (i) and $C^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) \cong C^{I,J}(M)$ by (ii), we have $C^{I,J}(M) \cong C^{I,J}(C^{I,J}(M))$.

iv) The first exact sequence of Corollary 3.5 gives rise to an exact sequence

$$\cdots \to H_i^{I,J}(H_1^{I,J}(M)) \to H_i^{I,J}(C^{I,J}(M)) \to H_i^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) \to \cdots.$$

By Lemma 2.4, $H_i^{I,J}(H_1^{I,J}(M))=0$ and $H_i^{I,J}(\bigcap_{\mathfrak{a}\in \tilde{W}(I,J)} \mathfrak{a}M) = H_i^{I,J}(M)$ for all $i \geq 1$. Therefore, we have $H_i^{I,J}(C^{I,J}(M)) \cong H_i^{I,J}(M)$ for all $i \geq 1$.

v) It follows from (iii) and Corollary 3.4.

Theorem 3.8.

Let $f: M \to M'$ be a continuous homomorphism of linearly compact $R-$modules such that $\Lambda_{I,J}(\operatorname{Ker} f) \cong \operatorname{Ker} f$ and $\Lambda_{I,J}(\operatorname{Coker} f) \cong \operatorname{Coker} f$. Let $\varphi: K \to M'$ be an other homomorphism of linearly compact $R-$modules. Then

i) $C^{I,J}(M) \cong C^{I,J}(M')$;

ii) *There is a unique homomorphism $\phi: C^{I,J}(K) \to M$ such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \uparrow_{\phi} & & \uparrow_{\varphi} \\ C^{I,J}(K) & \xrightarrow{\eta_K} & K \end{array}$$

is commutative;

iii) *If* $\varphi : K \to M'$ *and* $\eta_M : C^{I,J}(M) \to M$ *are both isomorphisms, then the homomorphism* ϕ *of part (ii) is also an isomorphism.*

Proof.

i) Applying the functor $C^{I,J}(-)$ to the exact sequences

$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$

$0 \to \operatorname{Im} f \to M' \to \operatorname{Coker} f \to 0.$

we have the following exact sequences

$C^{I,J}(\operatorname{Ker} f) \to C^{I,J}(M) \to C^{I,J}(\operatorname{Im} f) \to 0$

$C_1^{I,J}(\operatorname{Coker} f) \to C^{I,J}(\operatorname{Im} f) \to C^{I,J}(M') \to C^{I,J}(\operatorname{Coker} f) \to 0.$

By the hypothesis, $\Lambda_{I,J}(\operatorname{Ker} f) \cong \operatorname{Ker} f$ and $\Lambda_{I,J}(\operatorname{Coker} f) \cong \operatorname{Coker} f$. Then we have $C^{I,J}(\operatorname{Ker} f) = C^{I,J}(\operatorname{Coker} f) = 0$ by Theorem 3.7 (i). From Theorem 3.3 (i) and Lemma 2.4, we have

$C_1^{I,J}(\operatorname{Coker} f) \cong H_2^{I,J}(\operatorname{Coker} f) \cong H_2^{I,J}(\Lambda_{I,J}(\operatorname{Coker} f)) = 0.$

Therefore, $C^{I,J}(M) \cong C^{I,J}(\operatorname{Im} f) \cong C^{I,J}(M').$

ii) We have a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & M' & \xleftarrow{\varphi} & K \\ \uparrow{\scriptstyle \eta_M} & & \uparrow{\scriptstyle \eta_{M'}} & & \uparrow{\scriptstyle \eta_K} \\ C^{I,J}(M) & \xrightarrow{C^{I,J}(f)} & C^{I,J}(M') & \xleftarrow{C^{I,J}(\varphi)} & C^{I,J}(K). \end{array}$$

By (i), $C^{I,J}(f)$ is isomorphism. Let $\phi = \eta_M \circ C^{I,J}(f)^{-1} \circ C^{I,J}(\varphi)$. Then

$f \circ \phi = f \circ \eta_M \circ C^{I,J}(f)^{-1} \circ C^{I,J}(\varphi) = \eta_{M'} \circ C^{I,J}(\varphi) = \varphi \circ \eta_K.$

Assume that there is a homomorphism $\phi' : C^{I,J}(K) \to M$ such that $f \circ \phi' = \varphi \circ \eta_K$. We have the following commutative diagram

$$\begin{array}{ccccc} M & \xleftarrow{\eta_M} & C^{I,J}(M) & \xrightarrow{C^{I,J}(f)} & C^{I,J}(M') \\ \uparrow{\scriptstyle \phi'} & & \uparrow{\scriptstyle C^{I,J}(\phi')} & & \uparrow{\scriptstyle C^{I,J}(\varphi)} \\ C^{I,J}(K) & \xleftarrow{\eta_{C^{I,J}(K)}} & C^{I,J}(C^{I,J}(K)) & \xrightarrow{C^{I,J}(\eta_K)} & C^{I,J}(K), \end{array}$$

where $\eta_{C^{I,J}(K)} = C^{I,J}(\eta_K)$ is isomorphism by Theorem 3.7 (iii). It follows

$\phi' = \eta_M \circ C^{I,J}(\phi') \circ C^{I,J}(\eta_K)^{-1} = \eta_M \circ C^{I,J}(f)^{-1} \circ C^{I,J}(\varphi) = \phi.$

So ϕ is unique.

iii) Since φ is an isomorphism, $C^{I,J}(\varphi)$ is also an isomorphism. From (ii), we have $\phi = \eta_M \circ C^{I,J}(f)^{-1} \circ C^{I,J}(\varphi)$, hence ϕ is an isomorphism.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

REFERENCES

[1] T.T. Nam, "Ideal co-transforms of linearly compact modules," *East - West J. of Mathematics* vol. 6, no 2, pp. 173-183, 2004.

[2] M. P. Brodmann, and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications.* Cambridge University Press, 1998.

[3] T. T. Nam, and N. M. Tri, "Genreralized ideal transforms," *Stud. Sci. Math. Hung.*, 51, pp.67-82, 2014

[4] N. M. Tri, and T. T. Nam, "Ideal transfroms with respect to a pair of ideals," *Acta Math. Vietnam*, vol 42, pp. 701-715, 2017.

[5] T. T. Nam, and D. N. Yen, "Local homology with respect to a pair of ideal (I, J)," reprint.

[6] R. Takahashi, Y. Yoshino, and T. Yoshizawa, "Local cohomology based on a nonclosed support defined by a pair of ideals," *J. Pure Appl. Algebra*, 213, pp. 582-600, 2009.

[7] I. G. Macdonald, "Duality over complete local rings," *Topology 1*, pp. 213-235, 1962.

[8] C. U. Jensen, *Les Foncteurs Dérivés de* $\underleftarrow{\lim}_{t}$ *et leurs Applications en Théorie des Modules*. Springer-Verlag, Berlin-Heidelberg-New York, 1997.

[9] N. T. Cuong, and T. T. Nam, "A Local homology theory for linearly compact modules," *J. Algebra*, 319, pp. 4712-4737, 2008.