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A CAUCHY PROBLEM FOR THE ASYMMETRIC PARABOLIC EQUATION IN POLAR COORDINATES WITH THE PERTURBED DIFFUSIVITY

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ABSTRACT

The inverse problem for the heat equation plays an important area of study and application. Up to now, the backward heat problem (BHP) in Cartesian coordinates has been arisen in many articles, but the BHP in different domains such as polar coordinates, cylindrical one or spherical one is rarely considered. This paper's purpose is to investigate the BHP on a disk, especially, the problem is associated with the perturbed diffusivity and the space-dependent heat source. In order to solve the problem, the authors apply the separation of variables method, associated with the Bessel's equation and Bessel's expansion. Based on the exact solution, the regularized solution is constructed by using the modified quasi-boundary value method. As a result, a Holder type of convergence rate has been obtained. In addition, a numerical experiment is given to illustrate the flexibility and effectiveness of the used method.

Keywords: backward heat problem, modified quasi-boundary value method, polar coordinates, ill-posed problem.

1. Introduction

The consideration of the forward heat problem aims at predicting the temperature distribution of a body at a future time from the initial temperature, boundary conditions. On the contrary, the aim of the backward heat problem (BHP) is to determine the initial temperature from the final data. The BHP plays a vital role in practical applications such as image deblurring, mathematical finance, hydrologic inversion, mechanics of continuous media, so forth. In hydrologic inversion, by reconstructing the contaminant history, sources of groundwater pollution are sought and this problem is described by a simple form of the well-known advection-convection equation $u_t - b(t)\Delta u = f(x,t)$ (see in (Atmadja & Bagtzoglou, 2003; Quan et al., 2011; Trong & Tuan, 2008). Very recently, in (Tuan et al., 2016), Tuan et al. have considered the problem which is more general than the problem in (Atmadja & Bagtzoglou, 2003; Quan et al., 2011; Trong & Tuan, 2008,).

$$u_t - b(t)L(u) = f(x,t), (x,t) \in \Omega \times (0,T), \tag{1}$$

$$u|_{\partial\Omega} = 0, 0 < t < T, \tag{2}$$

$$u(x,T) = g(x), x \in \Omega, \tag{3}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $b(t)$, $g(x)$, $f(x,t)$ are known functions, and L is a symmetric elliptic operator. Then, the authors applied the filter regularization method to get an approximate solution and obtained the Holder type of error estimate (see in (Tuan et al., 2016)).

Although, there are many works related to the BHP in Cartesian coordinates (Fu et al., 2007; Quan et al., 2011; Trong et al., 2009; Trong & Tuan, 2008; Tuan & Trong, 2011); Tuan et al., 2016), the studies, associated with the BHP in polar coordinates, are considered rarely. Recently, an axisymmetric backward heat equation on a disk has been investigated by Cheng W. and Fu C. L. In the papers (Cheng & Fu, 2009; Cheng & Fu, 2010), Cheng and Fu used the spectral method and the modified Tikhonov method for regularizing the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}, & 0 < r \le r_0, \quad 0 < t < T, \\ u(r,T) = \varphi(r), & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ |u(0,t)| < \infty, & 0 \le t \le T. \end{cases} \tag{4}$$

In addition, they got the error estimate of logarithmic type which is presented in (Cheng & Fu, 2009; Cheng & Fu, 2010). It is remarkable that the measured data $\varphi(r)$ in problem (4) is radially symmetric or axisymmetric, i.e., it depends only on the radius r but not on θ. Consequently, the solution of problem (4) does not depend on θ. Otherwise, in practical engineering, the measured data is not always radially symmetric or axisymmetric. Furthermore, the initial temperature not only depends on the final temperature distribution but also depends on the heat source. Nevertheless, the papers (Cheng & Fu, 2009; Cheng & Fu, 2010) are mainly devoted to the homogeneous case of the heat source. To generalize the problem (4), the authors considered the problem of finding the initial temperature distribution of the asymmetrically nonhomogeneous parabolic equation in polar coordinates in (Triet et al., 2019). By the modified quasi-boundary method (MQBV) in (Quan et al., 2011), the authors constructed the approximated solution and obtained its convergence of Holder type. However, in (Triet et al., 2019), the *a priori* assumption on the exact solution in (Triet et al., 2019) must depend on a class of Gevrey spaces. In this paper, the authors improve this weak point by assuming the condition *(H_1)* in Section 3.

Besides that, in reality, the heat coefficient depends on material of the body, but an arbitrary body is not completely homogeneous. Therefore, the heat coefficient can be perturbed. Motivated by these reasons, in this article, let T be a positive number, the authors are interested in the problem of determining the temperature distribution $u(r,\theta,t),(r,\theta,t)\in(0,a)\times(0,2\pi)\times(0,T)$ satisfying the following problem

$$u_t = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)+q(r,\theta), 0<r<a, 0<\theta<2\pi, 0<t<T, \quad (5)$$

$$u(a,\theta,t)=0, 0<\theta<2\pi, 0<t<T, \quad (6)$$

$$|u(0,\theta,t)|<\infty, 0<\theta<2\pi, 0<t<T, \quad (7)$$

and the final temperature distribution

$$u(r,\theta,T)=f(r,\theta),\ 0<\theta<2\pi, 0<r<a, \quad (8)$$

where $f(\cdot,\theta)$, $q(\cdot,\theta)\in L^2\left[[0;a];r\right]$, $c\in\mathrm{R}$ are the final temperature, the heat source and the diffusivity, respectively. In reality, the final temperature f, the heat source q and the diffusivity c are obtained by measurement, so there are always errors. Assume that the exact data (f,c,q) and the measured data $(f_\varepsilon,c_\varepsilon,q_\varepsilon)$ satisfy

$$\left\|f^\varepsilon(\cdot,\theta)-f(\cdot,\theta)\right\|_2\le\varepsilon, |c_\varepsilon-c|\le\varepsilon, \left\|q^\varepsilon(\cdot,\theta)-q(\cdot,\theta)\right\|_2\le\varepsilon, \quad (9)$$

and ε is a noise level from a measurement. In our knowledge, the works for a Cauchy problem for the parabolic equation on a disk are quite scare and even there is no result dealt with the perturbed case of the diffusivity and the heat source. It is well-known that the problem (5)-(8) is ill-posed. This means that its solution may not exist, and if it exists, it does not depend continuously on the given data. In practical engineering, to get the initial data, it is a must to use equipment to measure, this leads to the error between exact data and measured data. Thus, small error on the measured data may lead to solutions with large errors. This makes the numerical computation difficult, so an appropriate regularization process is required in order to get a stable solution. In this paper, the MQBV is employed to construct the regularized solution for the problem (5)-(8). An appropriate "corrector term" is added into the boundary condition to get a regularized solution. Furthermore, a numerical example is given to prove the effectiveness of the used method.

The rest of the paper is organized as follows: In Section 2, some definitions and propositions are given to solve the problem (5)-(8). In Section 3, the authors propose the regularized solutions for the problem (5)-(8) and obtain the error estimate between the regularized solutions and the exact solution. Finally, a numerical experiment is presented to illustrate the main results in Section 4. Eventually, there is a conclusion in Section 5.

2. Some Definitions and Propositions

Throughout this paper, the authors denote the space of Lebesgue measurable functions f with weight r on $[0;a]$ by $L^2\left[[0;a];r\right]$ through the following definition.

Definition 2.1.

Let $a > 0$, *we define* $L^2\left[[0;a];r\right] = \{ f : [0;a] \to \mathrm{R} \mid f$ *is Lebesgue measurable with weigh* r *on* $[0;a]\}$. *We can see that the above space is normal space with the norm as follows*

$$\| f \|_2 = \left(\int_0^a r \left| f(r) \right|^2 dr \right)^{\frac{1}{2}}, \ \textit{for} \ f \in L^2\left[[0;a];r\right].$$

From here on, definition and some propositions are restated with the helps of the references (Frank, 1958; Watson, 1944).

Definition 2.2.

Let m *be a non-negative integer. Then we have Bessel functions of the* 1^{st} − *kind of order* m

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2} \right)^{2k+m}, \tag{10}$$

and Bessel functions of the 2^{nd} − *kind of order* m

$$Y_m(x) = \frac{2}{\pi} \left\{ \ln\left(\frac{x}{2} \right) + \gamma \right\} J_m(x) - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-m}$$
$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \{ h_k + h_{m+k} \} \left(\frac{x}{2} \right)^{2k+m}, \tag{11}$$

in which

$$\gamma = \lim_{n \to +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \ \textit{is Euler's constant,}$$

$$h_k = \sum_{i=1}^{k} \frac{1}{i}.$$

Proposition 2.3. *Let* m *be a non-negative integer, the Bessel's equation of order* m *is defined as follows*

$$x^2 y'' + x y' + (x^2 - m^2) y = 0, \ \ x > 0, \tag{12}$$

then we have the general solution of equation (12) is

$$y(x) = c_1 J_m(x) + c_2 Y_m(x),$$

where $J_m(x)$ *and* $Y_m(x)$ *is defined by (10) and (11), respectively.*

Proposition 2.4. *The equation* $J_m(x) = 0$ *has infinite real roots* $\left\{ x_n^{(m)} \right\}_{n \in \mathrm{Z}^+}$ *satisfying*

$$0 < x_1^{(m)} < x_2^{(m)} < \ldots < x_n^{(m)} < \ldots$$

and $\lim_{n\to\infty} x_n^{(m)} = +\infty.$

3. Regularizing and main results

By using the method of separation of variables, it can be clear to find out the exact solution u of the problem (5)-(8) corresponding to the exact data (f, c) as follows

$$u(f,c,q)(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r) u_{mn}(f,c,q)(\theta,t), \tag{13}$$

where

$$u_{mn}(f,c,q)(\theta,t) = A_{mn}[f,c,q](t)\cos m\theta + B_{mn}[f,c,q](t)\sin m\theta, \tag{14}$$

$$A_{mn}[f,c,q](t) = \left(a_{mn}[f] - \frac{a_{mn}[q]}{c^2\lambda_{mn}^2}\right)\exp\left\{c^2\lambda_{mn}^2(T-t)\right\} + \frac{a_{mn}[q]}{c^2\lambda_{mn}^2}, \tag{15}$$

$$a_{mn}[f] = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos(m\theta) J_m(\lambda_{mn} r) r d\theta dr,$$

$$a_{mn}[q] = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} q(r,\theta)\cos(m\theta) J_m(\lambda_{mn} r) r d\theta dr,$$

$$B_{mn}[f,c,q](t) = \left(b_{mn}[f] - \frac{b_{mn}[q]}{c^2\lambda_{mn}^2}\right)\exp\left\{c^2\lambda_{mn}^2(T-t)\right\} + \frac{b_{mn}[q]}{c^2\lambda_{mn}^2}, \tag{16}$$

$$b_{mn}[q] = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} q(r,\theta)\sin(m\theta) J_m(\lambda_{mn} r) r d\theta dr,$$

$$b_{mn}[f] = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin(m\theta) J_m(\lambda_{mn} r) r d\theta dr,$$

J_m is the Bessel function order of m,

$\lambda_{mn} = \frac{\alpha_{mn}}{a}, \alpha_{mn}$ is the nth positive zero of J_m.

In fact, to gain the final data and the diffusivity, the measured equipment is used. Therefore, there will appear the error of the exact data and the measured data. By analyzing the exact solution (13), it is shown that the data error can be arbitrarily amplified by the familiar "heat kernel" $\exp\left\{c^2\lambda_{mn}^2(T-t)\right\}$. Thus, it causes the ill-posedness of the problem (5)-(8). To construct a regularized solution for (13), the modified quasi-boundary value method is applied. The main idea is to replace the term $\exp\left\{c^2\lambda_{mn}^2(T-t)\right\}$ by a "stability term" $\frac{\exp\left\{-c_\varepsilon^2\lambda_{mn}^2 t\right\}}{\alpha(\varepsilon)c_\varepsilon^2\lambda_{mn}^2 + \exp\left\{-c_\varepsilon^2\lambda_{mn}^2 T\right\}}$ to get a stable solution. In particular, the authors formulate the regularized solution u^ε corresponding to the measured data $\left(f_\varepsilon, c_\varepsilon, q_\varepsilon\right)$ as follows

$$u^\varepsilon\left(f_\varepsilon, c_\varepsilon, q_\varepsilon\right)(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\lambda_{mn} r\right) u_{mn}^\varepsilon\left(f_\varepsilon, c_\varepsilon, q_\varepsilon\right)(\theta,t), \tag{17}$$

where

$$u_{mn}^\varepsilon\left(f_\varepsilon, c_\varepsilon, q_\varepsilon\right)(\theta,t) = A_{mn}^\varepsilon\left[f_\varepsilon, c_\varepsilon, q_\varepsilon\right](t)\cos m\theta + B_{mn}^\varepsilon\left[f_\varepsilon, c_\varepsilon, q_\varepsilon\right](t)\sin m\theta, \tag{18}$$

$$A_{mn}^\varepsilon\left[f_\varepsilon, c_\varepsilon, q_\varepsilon\right](t) = \left(a_{mn}\left[f_\varepsilon\right] - \frac{a_{mn}\left[q_\varepsilon\right]}{c_\varepsilon^2\lambda_{mn}^2}\right)\frac{\exp\left\{-c_\varepsilon^2\lambda_{mn}^2 t\right\}}{\alpha(\varepsilon)c_\varepsilon^2\lambda_{mn}^2 + \exp\left\{-c_\varepsilon^2\lambda_{mn}^2 T\right\}} + \frac{a_{mn}\left[q_\varepsilon\right]}{c_\varepsilon^2\lambda_{mn}^2},$$

$$B_{mn}^\varepsilon\left[f_\varepsilon, c_\varepsilon, q_\varepsilon\right](t) = \left(b_{mn}\left[f_\varepsilon\right] - \frac{b_{mn}\left[q_\varepsilon\right]}{c_\varepsilon^2\lambda_{mn}^2}\right)\frac{\exp\left\{-c_\varepsilon^2\lambda_{mn}^2 t\right\}}{\alpha(\varepsilon)c_\varepsilon^2\lambda_{mn}^2 + \exp\left\{-c_\varepsilon^2\lambda_{mn}^2 T\right\}} + \frac{b_{mn}\left[q_\varepsilon\right]}{c_\varepsilon^2\lambda_{mn}^2},$$

$$a_{mn}\left[f_\varepsilon\right] = \frac{2}{\pi a^2 J_{m+1}^2\left(\alpha_{mn}\right)}\int_0^a\int_0^{2\pi} f_\varepsilon(r,\theta)\cos(m\theta) J_m\left(\lambda_{mn} r\right) r\,d\theta\,dr,$$

$$a_{mn}\left[q_\varepsilon\right] = \frac{2}{\pi a^2 J_{m+1}^2\left(\alpha_{mn}\right)}\int_0^a\int_0^{2\pi} q_\varepsilon(r,\theta)\cos(m\theta) J_m\left(\lambda_{mn} r\right) r\,d\theta\,dr,$$

$$b_{mn}\left[f_\varepsilon\right] = \frac{2}{\pi a^2 J_{m+1}^2\left(\alpha_{mn}\right)}\int_0^a\int_0^{2\pi} f_\varepsilon(r,\theta)\sin(m\theta) J_m\left(\lambda_{mn} r\right) r\,d\theta\,dr,$$

$$b_{mn}\left[q_\varepsilon\right] = \frac{2}{\pi a^2 J_{m+1}^2\left(\alpha_{mn}\right)}\int_0^a\int_0^{2\pi} q_\varepsilon(r,\theta)\sin(m\theta) J_m\left(\lambda_{mn} r\right) r\,d\theta\,dr,$$

and $\alpha(\varepsilon)$ is regularization parameter such that $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$. For the brief, it can be denoted that $\alpha(\varepsilon) = \alpha$. Without loss of generality, it can be assumed that $c_\varepsilon \geq c_{ex}$.

Lemma 3.1. *For $0 < \alpha < T, a > 0$, we have the following inequality*

$$\frac{1}{\alpha a + \exp\{-aT\}} \leq \frac{T}{\alpha}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}.$$

Lemma 3.2. *Let $0 \leq t \leq s \leq T$, $0 < \alpha < T$ and $a > 0$, then we get the following inequalities*

i. $$\frac{\exp\{(s-t-T)a\}}{\alpha a + \exp\{-aT\}} \leq \tilde{T}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t-s}{T}},$$

ii. $$\frac{\exp\{-ta\}}{\alpha a + \exp\{-aT\}} \leq \tilde{T}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1},$$

in which $\tilde{T} = \max\{1;T\}$.

It can be seen that the proofs of these above lemmas in (Quan et al., 2011).

Lemma 3.3. *Let* $a > 0$, $0 < t < T$, $0 < \alpha < \min\{1;T\}$, $c_\varepsilon > c > 0$ *and*

$$\varphi(x) = \frac{\exp\{-atx\}}{\alpha ax + \exp\{-aTx\}} \text{ for } x > 0$$

We have the following inequality

$$\left|\varphi(c_\varepsilon) - \varphi(c)\right| \le \left[2a\tilde{T}^2\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1} + \frac{\tilde{T}^2}{c}\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right](c_\varepsilon - c),$$

in which $\tilde{T} = \max\{1;T\}$.

Proof. It is easy to see that φ is continuous and derivative on $[c, c_\varepsilon]$. By using the Lagrange's theorem, we obtain $x_0 \in [c, c_\varepsilon]$ satisfying

$$\begin{aligned}
&\left|\varphi(c_\varepsilon) - \varphi(c)\right| \\
&\le \left|\varphi'(x_0)\right|(c_\varepsilon - c) \\
&\le \left|\frac{-at\exp\{-atx_0\}}{\alpha ax_0 + \exp\{-aTx_0\}} - \frac{\exp\{-atx_0\}\left(\alpha a - aT\exp\{-aTx_0\}\right)}{\left(\alpha ax_0 + \exp\{-aTx_0\}\right)^2}\right|(c_\varepsilon - c) \\
&\le \left[aT\tilde{T}\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1} + a\tilde{T}\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\left(\frac{1}{ax_0} + T\right)\right](c_\varepsilon - c) \\
&\le \left[2a\tilde{T}^2\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1} + \frac{\tilde{T}^2}{c}\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right](c_\varepsilon - c).
\end{aligned}$$

This completes the proof of Lemma 3.3.

Lemma 3.4. *Let* $a > 0$, $0 < t < T$, $0 < \alpha < \min\{1;T\}$, $c_\varepsilon > c > 0$ *and*

$$\psi(x) = \frac{1}{xa}\left(\frac{\exp\{-atx\}}{\alpha ax + \exp\{-aTx\}} - 1\right) \text{for } x > 0.$$

We get

$$\begin{aligned}
&\left|\psi(c_\varepsilon) - \psi(c)\right| \\
&\le \left[\frac{T}{c}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1} + \frac{\tilde{T}}{ac^2}\right](c_\varepsilon - c) + \left[\frac{2\tilde{T}^2}{c} + \frac{\tilde{T}^2}{ac^2}\right]\left(\alpha \ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}(c_\varepsilon - c).
\end{aligned}$$

Proof. By simple calculations, we deduce that

$$\psi'(x)=\frac{-1}{ax^2}\left(\frac{\exp\{-atx\}}{\alpha ax+\exp\{-aTx\}}-1\right)+\frac{1}{xa}\left(\frac{-at\exp\{-atx\}}{\alpha ax+\exp\{-aTx\}}\right.$$

$$\left.-\frac{\exp\{-atx\}\left(\alpha a-aT\exp\{-atx_0\}\right)}{\left(\alpha ax_0+\exp\{-aTx_0\}\right)^2}\right).$$

Similar to Lemma 3.1 - Lemma 3.3, we apply the Lagrange's theorem, we have $x_0\in[c,c_\varepsilon]$ satisfying

$$\left|\psi(c_\varepsilon)-\psi(c)\right|$$

$$\le\left|\frac{-1}{ax_0^2}\left(\frac{\exp\{-atx_0\}}{\alpha ax_0+\exp\{-aTx_0\}}-1\right)+\frac{1}{x_0a}\left(\frac{-at\exp\{-atx_0\}}{\alpha ax_0+\exp\{-aTx_0\}}\right.\right.$$

$$\left.\left.-\frac{\exp\{-atx_0\}\left(\alpha a-aT\exp\{-atx_0\}\right)}{\left(\alpha ax_0+\exp\{-aTx_0\}\right)^2}\right)\right|(c_\varepsilon-c)$$

$$\le\frac{1}{ax_0^2}\frac{\alpha ax_0+\exp\{-aTx_0\}-\exp\{-atx_0\}}{\alpha ax_0+\exp\{-aTx_0\}}(c_\varepsilon-c)$$

$$+\frac{1}{ac}\left[2a\overset{\square}{T}^2\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}+\frac{1}{c}\overset{\square}{T}^2\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right](c_\varepsilon-c)$$

$$\le\left[\frac{\alpha}{x_0}\frac{T}{\alpha}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}+\frac{\overset{\square}{T}}{ax_0^2}\right](c_\varepsilon-c)$$

$$+\left[\frac{2\overset{\square}{T}^2}{c}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}+\frac{\overset{\square}{T}^2}{ac^2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right](c_\varepsilon-c)$$

$$\le\left[\frac{T}{c}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}+\frac{\overset{\square}{T}}{ac^2}\right](c_\varepsilon-c)$$

$$+\left[\frac{2\overset{\square}{T}^2}{c}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}+\frac{\overset{\square}{T}^2}{ac^2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right](c_\varepsilon-c).$$

This completes the proof of Lemma 3.4.

In this paper, the authors require some assumptions on the exact solution and the heat source q as follows

(H_1): There exists a non-negative number A such that

$$\sup_{\theta\in[0;2\pi]}\left\{\left\|\frac{\partial}{\partial t}u\left(f,c,q\right)(\cdot,\theta,T)\right\|_2+\left\|\frac{\partial}{\partial t}u\left(f,c,q\right)(\cdot,\theta,0)\right\|_2+\left\|f(\cdot,\theta)\right\|_2+\left\|q(\cdot,\theta,t)\right\|_2\right\}\le A.$$

Theorem 3.1. *Let* $\left(f,c,q\right)$, $\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)\in L^2\left[\left[0;a\right];r\right]\times \mathrm{R}\times L^2\left[\left[0;a\right];r\right]$ *satisfy the condition* (9), $0<\varepsilon<\min\{1;T\}$ *and* $\alpha=\varepsilon$. *Assume that* $u\left(f,c,q\right)$ *and* $u^\varepsilon\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)$, *defined by (13) and (17), are corresponding to the exact data* $\left(f,c,q\right)$ *and the measured data* $\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)$, *respectively. For* $(\theta,t)\in\left[0;2\pi\right]\times\left[0;T\right]$, *we obtain the error estimate*

$$\left\|u^\varepsilon\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)(\cdot,\theta,t)-u\left(f,c,q\right)(\cdot,\theta,t)\right\|_2\le\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}N\left(\varepsilon,t\right),\qquad (19)$$

in which

$$N\left(\varepsilon,t\right)=2\overset{\square}{T}^2R(c)A\left(2+\frac{2}{\lambda_{0,1}^2}+\varepsilon^{1-\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}\right)$$

$$+\overset{\square}{T}\left(A+\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}\varepsilon^{1-\frac{t}{T}}+1\right),$$

$\lambda_{0,1}=\dfrac{\alpha_{0,1}}{a},\alpha_{0,1}$ *is the* $0th$ *positive zero of* J_1,

$R(c)=\max\left\{2c+1;\dfrac{2c+1}{c^2};\dfrac{1}{c^4}\right\}.$

Proof. By using the triangle inequality, we infer that

$$\left\|u^\varepsilon\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)(\cdot,\theta,t)-u\left(f,c,q\right)(\cdot,\theta,t)\right\|_2$$

$$\le\left\|u^\varepsilon\left(f_\varepsilon,c_\varepsilon,q_\varepsilon\right)(\cdot,\theta,t)-u^\varepsilon\left(f,c_\varepsilon,q\right)(\cdot,\theta,t)\right\|_2+\left\|u^\varepsilon\left(f,c_\varepsilon,q\right)(\cdot,\theta,t)-u^\varepsilon\left(f,c,q\right)(\cdot,\theta,t)\right\|_2$$

$$+\left\|u^\varepsilon\left(f,c,q\right)(\cdot,\theta,t)-u\left(f,c,q\right)(\cdot,\theta,t)\right\|_2,\qquad (20)$$

where

$$u^\varepsilon\left(f,c_\varepsilon,q\right)(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\lambda_{mn}r\right)u_{mn}^\varepsilon\left(f,c_\varepsilon,q\right)(\theta,t),\qquad (21)$$

$$u_{mn}^\varepsilon\left(f,c_\varepsilon,q\right)(\theta,t)=A_{mn}^\varepsilon\left[f,c_\varepsilon,q\right](t)\cos m\theta+B_{mn}^\varepsilon\left[f,c_\varepsilon,q\right](t)\sin m\theta,\qquad (22)$$

$$A_{mn}^\varepsilon\left[f,c_\varepsilon,q\right](t)=\left(a_{mn}\left[f\right]-\frac{a_{mn}\left[q\right]}{c_\varepsilon^2\lambda_{mn}^2}\right)\frac{\exp\left\{-c_\varepsilon^2\lambda_{mn}^2t\right\}}{\alpha c_\varepsilon^2\lambda_{mn}^2+\exp\left\{-c_\varepsilon^2\lambda_{mn}^2T\right\}}+\frac{a_{mn}\left[q\right]}{c_\varepsilon^2\lambda_{mn}^2},$$

$$B_{mn}^{\varepsilon}\left[f,c_{\varepsilon},q\right](t)=\left(b_{mn}\left[f\right]-\frac{b_{mn}\left[q\right]}{c_{\varepsilon}^{2}\lambda_{mn}^{2}}\right)\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}+\frac{b_{mn}\left[q\right]}{c_{\varepsilon}^{2}\lambda_{mn}^{2}},$$

$$u^{\varepsilon}\left(f,c,q\right)(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_{m}\left(\lambda_{mn}r\right)u_{mn}^{\varepsilon}\left(f,c,q\right)(\theta,t),\tag{23}$$

$$u_{mn}^{\varepsilon}\left(f,c,q\right)(\theta,t)=A_{mn}^{\varepsilon}\left[f,c,q\right](t)\cos m\theta+B_{mn}^{\varepsilon}\left[f,c,q\right](t)\sin m\theta,\tag{24}$$

$$A_{mn}^{\varepsilon}\left[f,c,q\right](t)=\left(a_{mn}\left[f\right]-\frac{a_{mn}\left[q\right]}{c^{2}\lambda_{mn}^{2}}\right)\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}+\frac{a_{mn}\left[q\right]}{c^{2}\lambda_{mn}^{2}},$$

$$B_{mn}^{\varepsilon}\left[f,c,q\right](t)=\left(b_{mn}\left[f\right]-\frac{b_{mn}\left[q\right]}{c^{2}\lambda_{mn}^{2}}\right)\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}+\frac{b_{mn}\left[q\right]}{c^{2}\lambda_{mn}^{2}},$$

From (17), (21), Lemma 3.1 and Lemma 3.2, we estimate

$$\left\|u^{\varepsilon}\left(f_{\varepsilon},c_{\varepsilon},q_{\varepsilon}\right)(\cdot,\theta,t)-u^{\varepsilon}\left(f,c_{\varepsilon},q\right)(\cdot,\theta,t)\right\|_{2}$$

$$\leq\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}C_{mn}\left[f_{\varepsilon}-f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{c_{\varepsilon}^{2}\lambda_{mn}^{2}}\left(1-\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}\right)C_{mn}\left[q_{\varepsilon}-q\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$\leq\square T\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[f_{\varepsilon}-f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}C_{mn}\left[q_{\varepsilon}-q\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$\leq\square T\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\left\|f^{\varepsilon}\left(\cdot,\theta\right)-f\left(\cdot,\theta\right)\right\|_{2}+T\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}\left\|q^{\varepsilon}\left(\cdot,\theta\right)-q\left(\cdot,\theta\right)\right\|_{2}$$

$$\leq\square T\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon+T\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}\varepsilon\tag{25}$$

where

$$C_{mn}\left[g\right](\theta)=a_{mn}\left[g\right]\cos m\theta+b_{mn}\left[g\right]\sin m\theta.$$

From (22), (24), this implies that

$$u_{mn}^{\varepsilon}\left(f,c_{\varepsilon},q\right)(\theta,t)-u_{mn}^{\varepsilon}\left(f,c,q\right)(\theta,t)$$

$$= C_{mn}\left[f\right](\theta)\left(\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}-\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}\right)$$

$$+C_{mn}\left[q\right](\theta)\left[\frac{1}{c^{2}\lambda_{mn}^{2}}\left(\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}-1\right)\right.$$

$$\left.-\frac{1}{c_{\varepsilon}^{2}\lambda_{mn}^{2}}\left(\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}-1\right)\right]$$

By applying Lemma 3.3 and Lemma 3.4, we infer that

$$\left\|u^{\varepsilon}\left(f,c_{\varepsilon},q\right)(\cdot,\theta,t)-u^{\varepsilon}\left(f,c,q\right)(\cdot,\theta,t)\right\|_{2}$$

$$\leq\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right.$$

$$\left.\times\left(\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}-\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}\right)\right\|_{2}$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[q\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\left[\frac{1}{c^{2}\lambda_{mn}^{2}}\left(\frac{\exp\left\{-c^{2}\lambda_{mn}^{2}t\right\}}{\alpha c^{2}\lambda_{mn}^{2}+\exp\left\{-c^{2}\lambda_{mn}^{2}T\right\}}-1\right)\right.\right.$$

$$\left.\left.-\frac{1}{c_{\varepsilon}^{2}\lambda_{mn}^{2}}\left(\frac{\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}t\right\}}{\alpha c_{\varepsilon}^{2}\lambda_{mn}^{2}+\exp\left\{-c_{\varepsilon}^{2}\lambda_{mn}^{2}T\right\}}-1\right)\right]\right\|_{2}$$

$$\leq\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right.$$

$$\left.\times\left(2\lambda_{mn}^{2}\tilde{T}^{2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}+\frac{\tilde{T}^{2}}{c^{2}}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right)\left(c_{\varepsilon}^{2}-c^{2}\right)\right\|_{2}$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[q\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\left[\frac{T}{c^{2}}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}+\frac{\tilde{T}}{\lambda_{mn}^{2}c^{4}}+\right.\right.$$

$$\leq 2\tilde{T}^{2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon\left(2c+1\right)\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}^{2}C_{mn}\left[f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$+\tilde{T}^{2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon\frac{2c+1}{c^{2}}\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[f\right](\theta)J_{m}\left(\lambda_{mn}\cdot\right)\right\|_{2}$$

$$+\left[\frac{T}{c^2}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}+\frac{\tilde{T}}{\lambda_{0,1}^2c^4}+\frac{2\tilde{T}^2}{c^2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}+\frac{\tilde{T}^2}{\lambda_{0,1}^2c^4}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\right]$$

$$\times\varepsilon(2c+1)\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}C_{mn}[q](\theta)J_m(\lambda_{mn}\cdot)\right\|_2$$

$$\leq 2\tilde{T}^2\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon R(c)\left\|\frac{\partial}{\partial t}u(f,c,q)(\cdot,\theta,T)\right\|_2+\tilde{T}^2\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon R(c)\left\|f(\cdot,\theta)\right\|_2$$

$$+\left[R(c)T\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}\varepsilon+\frac{\tilde{T}}{\lambda_{0,1}^2}R(c)\varepsilon+R(c)2\tilde{T}^2\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon\right.$$

$$\left.+\frac{R(c)\tilde{T}^2}{\lambda_{0,1}^2}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon\right]\left\|q(\cdot,\theta,t)\right\|_2$$

$$\leq 2\tilde{T}^2R(c)A\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon$$

$$+2\tilde{T}^2R(c)A\left[\left(\ln\left(\frac{T}{\alpha}\right)\right)^{-1}\varepsilon+\frac{\varepsilon}{\lambda_{0,1}^2}+\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\varepsilon\left(1+\frac{1}{\lambda_{0,1}^2}\right)\right]. \quad (26)$$

From (13), (23), we obtain

$$\left\|u^{\varepsilon}(f,c,q)(\cdot,\theta,t)-u(f,c,q)(\cdot,\theta,t)\right\|_2$$

$$=\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha c^2\lambda_{mn}^2\exp\{-c^2\lambda_{mn}^2t\}}{\alpha c^2\lambda_{mn}^2+\exp\{-c^2\lambda_{mn}^2T\}}\exp\{c^2\lambda_{mn}^2T\}C_{mn}[f](\theta)J_m(\lambda_{mn}\cdot)\right\|_2$$

$$\leq\alpha\tilde{T}\left(\alpha\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\left\|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}c^2\lambda_{mn}^2\exp\{c^2\lambda_{mn}^2T\}C_{mn}[f](\theta)J_m(\lambda_{mn}\cdot)\right\|_2$$

$$=\tilde{T}\alpha^{\frac{t}{T}}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}\left\|\frac{\partial}{\partial t}u(f,c)(\cdot,\theta,0)\right\|_2$$

$$\leq\tilde{T}\alpha^{\frac{t}{T}}\left(\ln\left(\frac{T}{\alpha}\right)\right)^{\frac{t}{T}-1}A. \quad (27)$$

By combining (25), (26) and (27), let $\alpha=\varepsilon,$ we get the following estimate

$$\left\|u^{\varepsilon}(f_{\varepsilon},c_{\varepsilon},q_{\varepsilon})(\cdot,\theta,t)-u(f,c,q)(\cdot,\theta,t)\right\|_2$$

$$\leq\tilde{T}\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}+T\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{-1}\varepsilon+2\tilde{T}^2R(c)A\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}$$

$$+2T^2R(c)A\left[\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{-1}\varepsilon+\frac{\varepsilon}{\lambda_{0,1}^2}+\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}\left(1+\frac{1}{\lambda_{0,1}^2}\right)\right]$$

$$+T\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}A$$

$$\le \varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}\left(T+T\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}\varepsilon^{1-\frac{t}{T}}+2T^2R(c)A\left(2+\frac{1}{\lambda_{0,1}^2}\right)+TA\right)$$

$$+\varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}2T^2R(c)A\left(\varepsilon^{1-\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}+\frac{1}{\lambda_{0,1}^2}\right)$$

$$\le \varepsilon^{\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{t}{T}-1}N(\varepsilon),$$

in which

$$N(\varepsilon,t)=2T^2R(c)A\left(2+\frac{2}{\lambda_{0,1}^2}+\varepsilon^{1-\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}\right)+T\left(A+\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}\varepsilon^{1-\frac{t}{T}}+1\right).$$

We note that $\lim\limits_{\varepsilon\to 0}\varepsilon^{1-\frac{t}{T}}\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{\frac{-t}{T}}=0.$

This completes the proof of Theorem 3.1.

Remark 3.2. *If $t>0$, we see that the error estimate (19) is a Holder type. On the other hand, when $t=0$, the error estimate (19) becomes*

$$\left\|u^{\varepsilon}\left(f_{\varepsilon},c_{\varepsilon},q_{\varepsilon}\right)(\cdot,\theta,t)-u\left(f,c,q\right)(\cdot,\theta,t)\right\|_2\le\left(\ln\left(\frac{T}{\varepsilon}\right)\right)^{-1}N(\varepsilon,0), \quad (28)$$

in which

$$N(\varepsilon,0)=2T^2R(c)A\left(2+\frac{2}{\lambda_{0,1}^2}+\varepsilon\right)+T\left(A+\varepsilon+1\right)$$

The estimate (28) is a logarithmic type of convergence rate.

4. Numerical experiment

In this section, the authors consider the following problem

$$u_t=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)+q(r,\theta),0<r<1,0<\theta<2\pi,0<t<1, \quad (29)$$

$$u(1,\theta,t) = 0, 0 < \theta < 2\pi, 0 < t < 1, \tag{30}$$

$$u(r,\theta,1) = f(r,\theta), 0 < \theta < 2\pi, 0 < r < 1, \tag{31}$$

$$|u(0,\theta,t)| < \infty, 0 < \theta < 2\pi, 0 < t < 1, \tag{32}$$

in which $c = 10^{-1}$ and

$$f(r,\theta) = (1-r^2)r^2 \sin 2\theta, q(r,\theta) = (1-r^2) r \sin\theta. \tag{33}$$

From (13), we get the exact solution of problem (29)-(32) corresponding to the exact data (f,c,q)

$$u(f,c,q)(r,\theta,t) = \sum_{n=1}^{\infty} \left(J_2(\alpha_{2n} r) \frac{24 \sin 2\theta}{c^2 \alpha_{2n}^5 J_3(\alpha_{2n})} \exp\left\{c^2 \alpha_{2n}^2 (1-t)\right\} \right.$$
$$\left. + J_1(\alpha_{1n} r) \frac{16 \sin\theta}{\alpha_{1n}^3 J_2(\alpha_{1n})} \left(1 - \exp\left\{c^2 \alpha_{1n}^2 (1-t)\right\}\right) \right). \tag{34}$$

Next, we consider measured data $(f_\varepsilon, c_\varepsilon, q_\varepsilon)$ as follows

$$f_\varepsilon(r,\theta) = \left(1 + \frac{\varepsilon \cdot rand(\cdot)}{\sqrt{\pi}}\right) f(r,\theta),$$
$$q_\varepsilon(r,\theta) = \left(1 + \frac{\varepsilon \cdot rand(\cdot)}{\sqrt{\pi}}\right) q(r,\theta), \tag{35}$$
$$c_\varepsilon = c + \varepsilon \cdot rand(\cdot),$$

where $rand(\cdot) \sim N(0,1)$. From (33) and (35), it is easy to see that

$$\left\| f_\varepsilon(\cdot,\theta) - f(\cdot,\theta) \right\|_2 \le \varepsilon, \left\| q_\varepsilon(\cdot,\theta) - q(\cdot,\theta) \right\|_2 \le \varepsilon \text{ for all } \theta \in (0;2\pi), |c_\varepsilon - c| \le \varepsilon.$$

From (17), we get the regularized solution of the problem (29)-(32) corresponding to the measured data $(f_\varepsilon, c_\varepsilon, q_\varepsilon)$

$$u^\varepsilon(f_\varepsilon, c_\varepsilon, q_\varepsilon)(r,\theta,t)$$
$$= \left(1 + \frac{\varepsilon \cdot rand(\cdot)}{\sqrt{\pi}}\right) \sum_{n=1}^{\infty} \frac{24 \sin 2\theta}{\alpha_{2n}^3 J_3(\alpha_{2n})} J_2(\alpha_{2n} r) \frac{\exp\left\{-c_\varepsilon^2 \alpha_{2n}^2 t\right\}}{\varepsilon c_\varepsilon^2 \alpha_{2n}^2 + \exp\left\{-c_\varepsilon^2 \alpha_{2n}^2\right\}}$$
$$+ \left(1 + \frac{\varepsilon \cdot rand(\cdot)}{\sqrt{\pi}}\right) \sum_{n=1}^{\infty} J_1(\alpha_{1n} r) \frac{16 \sin\theta}{\alpha_{1n}^3 J_2(\alpha_{1n})} \left(1 - \frac{\exp\left\{-c_\varepsilon^2 \alpha_{2n}^2 t\right\}}{\varepsilon c_\varepsilon^2 \alpha_{2n}^2 + \exp\left\{-c_\varepsilon^2 \alpha_{2n}^2\right\}}\right). \tag{36}$$

Supporting by the Maple program, it is possible to approximate the exact solution (34) and the regularized solution (36) associated with first one hundred coefficients. Then the authors give the following tables which show the error estimates between the exact solution and the regularized solution corresponding to the data error $\varepsilon_i = 10^{-i}, i = \overline{1,3}$, respectively.

Table 1. *Errors between the exact solution and the regularized solutions in case* $\theta = \dfrac{\pi}{3}$

	$\left\| u^{\varepsilon}(f_\varepsilon, c_\varepsilon, q_\varepsilon)\left(\cdot, \frac{\pi}{3}, t\right) - u(f,c,q)\left(\cdot, \frac{\pi}{3}, t\right) \right\|_2$		
t	$\varepsilon_1 = 10^{-1}$	$\varepsilon_2 = 10^{-2}$	$\varepsilon_3 = 10^{-3}$
0	4.1736×10^{-2}	1.9801×10^{-2}	2.8734×10^{-3}
0.5	9.2835×10^{-3}	4.2019×10^{-3}	6.3828×10^{-4}

Table 2. *Errors between the exact solution and the regularized solutions in case* $\theta = \dfrac{5\pi}{4}$

	$\left\| u^{\varepsilon}(f_\varepsilon, c_\varepsilon, q_\varepsilon)\left(\cdot, \frac{5\pi}{4}, t\right) - u(f,c,q)\left(\cdot, \frac{5\pi}{4}, t\right) \right\|_2$		
t	$\varepsilon_1 = 10^{-1}$	$\varepsilon_2 = 10^{-2}$	$\varepsilon_3 = 10^{-3}$
0	1.1799×10^{-1}	3.4353×10^{-2}	4.4434×10^{-3}
0.5	3.7359×10^{-2}	9.1890×10^{-3}	1.1814×10^{-3}

Furthermore, there are some following graphs of the exact solution $u(f,c,q)$ and the regularized solutions $u^{\varepsilon_i}(f_\varepsilon, c_\varepsilon, q_\varepsilon)$, $i = 1,2,3$ (Figure 1, 2) at $t = 0$. Eventually, Figure 3 can visually present the exact solution $u(f,c,q)$ and the regularized solutions $u^{\varepsilon_i}(f_\varepsilon, c_\varepsilon, q_\varepsilon), i = 1...3$ at $r = 0.5, t = 0$ in the polar coordinates.

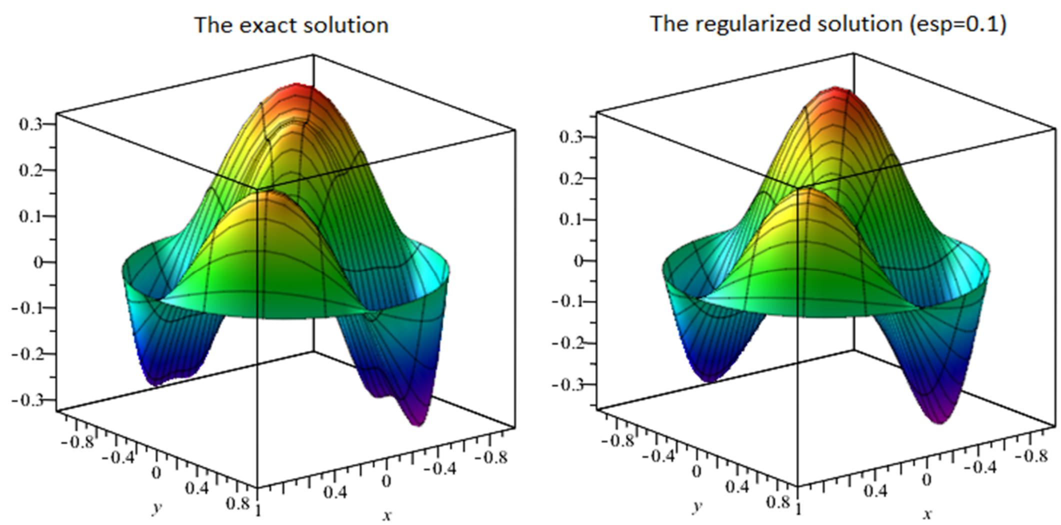

Figure 1. *The exact solution and the regularized solution corresponding to* ε_1

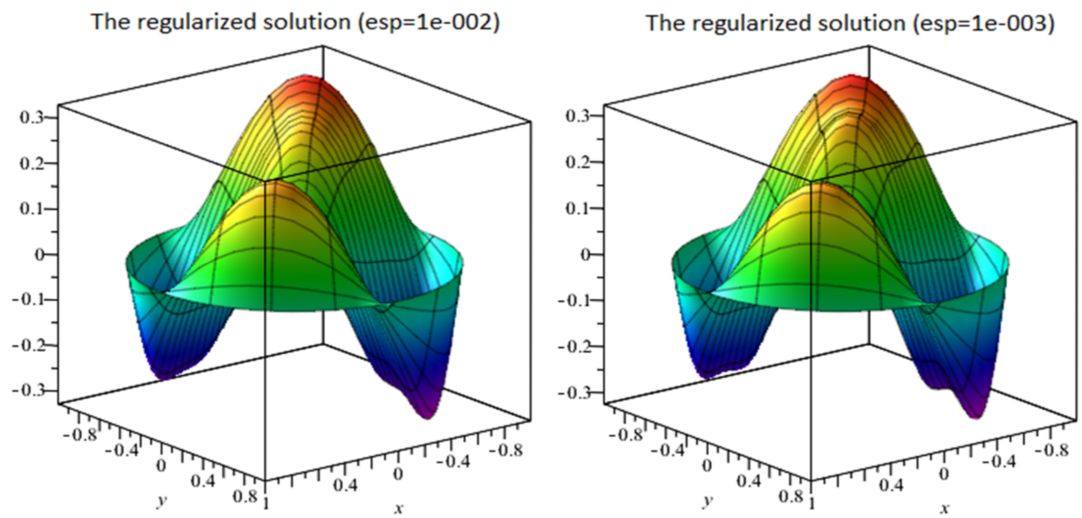

Figure 2. *The regularized solutions corresponding to* ε_2 *and* ε_3

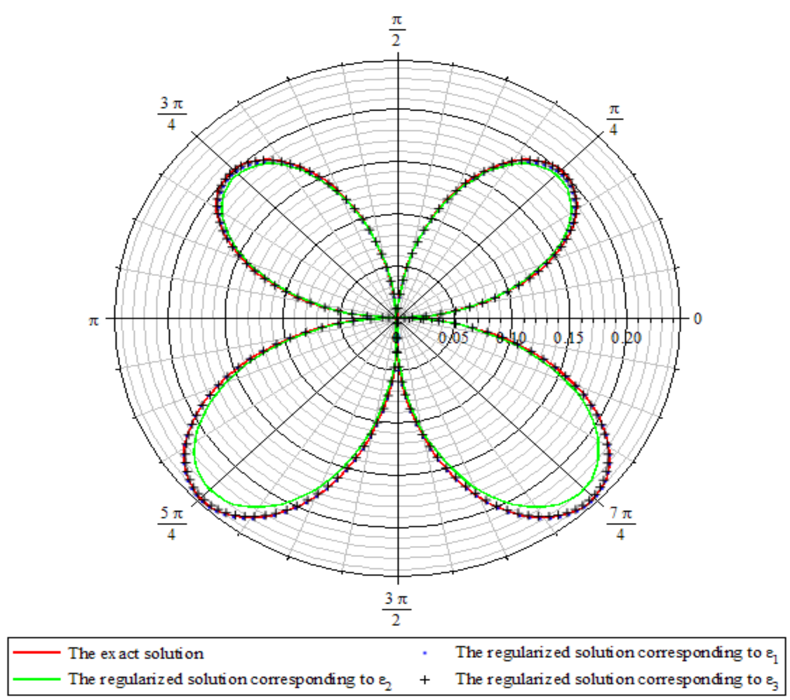

Figure 3. *The exact solution and the regularized solutions corresponding to* ε_i , $i = 1,...3$ *in the case* $r = 1$, $t = 0.5$ *in the polar coordinates*

5. Conclusion

In this paper, the authors considered a backward in time problem of the parabolic equation, associated with the perturbed diffusivity and the perturbed space-dependent heat source, in polar coordinates. Then the authors proposed the modified quasi-boundary value method (MQBV) to regularize this problem and obtained the error estimates between the exact solution and its regularized solutions in $L^2\left[\left[0;a\right];r\right]$ norm. Moreover, a numerical experiment shows that the used method is flexible and effective. In the future, desiring to research more generally on this problem, the authors will investigate the problem (5)-(8) with the time-dependent diffusivity or the space and time-dependent diffusivity.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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BÀI TOÁN CAUCHY CHO PHƯƠNG TRÌNH DẠNG PARABOLIC KHÔNG ĐỐI XỨNG TRONG TỌA ĐỘ CỰC VỚI HỆ SỐ KHUẾCH TÁN BỊ NHIỄU

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TÓM TẮT

Bài toán ngược cho phương trình nhiệt đóng vai trò quan trọng trong nghiên cứu và ứng dụng. Cho đến nay, bài toán nhiệt ngược thời gian (Backward heat problem - BHP) trong tọa độ Cartesian đã được nghiên cứu trong nhiều bài báo, nhưng BHP trong các tọa độ khác như tọa độ cực, tọa độ trụ hoặc tọa độ cầu lại hiếm khi được xem xét. Trong bài báo này, chúng tôi muốn nghiên cứu BHP trên một đĩa tròn, đặc biệt hơn, bài toán được xem xét liên hệ với hệ số khuếch tán bị nhiễu và nguồn nhiệt phụ thuộc vào không gian. Để giải quyết bài toán này, chúng tôi áp dụng phương pháp khai triển chuỗi Bessel. Dựa trên nghiệm chính xác, nghiệm chỉnh hóa được xây dựng bằng cách sử dụng phương pháp giá trị tựa biên. Kết quả là, chúng ta có được một ước lượng sai số hội tụ. Ngoài ra, một ví dụ số được đưa ra để minh họa tính hiệu quả của phương pháp.

Từ khóa: bài toán nhiệt ngược, phương pháp giá trị tựa biên, tọa độ cực, bài toán không chỉnh.

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