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A FAMILY OF ANALYTICALLY SOLVABLE SCHRÖDINGER EQUATIONS RELATED BY LEVI-CIVITA TRANSFORMATION

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ABSTRACT

Some two-dimensional problems in non-relativistic quantum mechanics can connect to each other by certain spatial transformations such as Levi-Civita transformation. This property allows forming a series of two-dimensional problems into an interrelated family. Starting from two related problems namely Coulomb plus harmonic oscillator and sextic double-well anharmonic oscillator potentials, such family is constructed via repeatedly applying Levi-Civita transformations. Obviously, this family contains various of exactly analytically solvable problems. The quasi-exact solution for each unknown member of this family is also obtained and systematically investigated.

Keywords: *quasi-exact solution,* analytically solvable, Levi-Civita transformation.

1. Introduction

It is known that some problems in non-relativistic quantum mechanics interestingly relate to each other by certain spatial transformations. This considerable property leads to arrange these related problems into a certain family and gives plenty of choices to solve them in both analytical and numerical approaches (Mavromatis, 1997, 1998a, 1998b). Particularly, Levi-Civita transformation (Levi-Civita, 1906) has been widely used as an efficient tool to connect the Schrödinger equations which describe two-dimensional motions under the influence of some central potentials, e.g Coulomb and harmonic oscillator potentials (Le & Nguyen, 1993) or Coulomb potential plus harmonic oscillator term (C+HO) and sextic double-well anharmonic oscillator (sextic DWAO) (Hoang-Do, Pham & Le, 2013). The former problem, which arises from two-dimensional hydrogen atom model under presence of a homogeneous magnetic field, has been analytically solved both in recurrence form (Taut, 1995) and in compact form (Le, Hoang & Le, 2017) while the exact analytical solution for later one is also obtained in compact form via the inspiration of one-dimension case (Le, Hoang & Le, 2018). Hence, repeatedly applying Levi-Civita transformation into C+HO and sextic DWAO problems can leads us to other problems whose analytical solution can be exactly obtained in compact form as the same as known ones shown by Le, Hoang & Le (2017, 2018). As an expected results, these problems combining with C+HO and sextic DWAO problems can form one of families whose members are interrelated by Levi-Civita problems; also, this family is exactly analytically solved in compact form.

On the other hand, Feranchuk-Komarov operator method whose idea is based on harmonic oscillator (Feranchuk et al., 2015) has been applied to achieve exact numerical solution for C+HO problem (Hoang-Do, Pham & Le, 2013) and sextic DWAO problem (Hoang-Do, 2016). After that, exact analytical solutions for these problems have also been used as good references to test how efficient Feranchuk-Komarov operator method is (Le, Hoang & Le, 2017, 2018). Similarly, other problems which belong to the expected family and their corresponding exact analytical solutions can also be used as other significant references for any numerical method including Feranchuk-Komarov operator method. Within these considerable reasons, the authors aim to construct this family of problems by repeatedly applying Levi-Civita transformation into C+HO and sextic DWAO problems. Another scope, which is also necessarily presented, is to perform exact analytical solution for these problems and to investigate these solutions systematically.

This work is constructed into three sections excluding the Introduction and the Conclusion. Section 2 obtains the general formula and model to construct the considered family of problems. Each member of this family is also listed in this section. Moreover, the brief about methodology to obtain quasi-exact solution for this family is described in this section. Next, results in quasi-exact solution are devoted in Section 3. Some further discussions about the relationship between topological property of Levi-Civita transformation and the number of quasi-exact solution in each generation of this family are also given in Section 4.

2. Formula

2.1. Levi-Civita transformation to relate two two-dimension Schrodinger equations

Levi-Civita transformation which was first introduced by Levi-Civita (1906) is a bilinear transformation which connects a two-dimension Euclean space $\mathbb{R}^2:(x,y)$ with another two-dimension Euclean space $\mathbb{R}^2:(u,v)$ such that the Euler identity is satisfied $u^2+v^2=\sqrt{x^2+y^2}$. The explicit expression of this transformation can be given as follow:

$$\mathrm{L}:(u,v)\to(x,y) \text{ such that } x=u^2-v^2 \text{ and } y=2uv\ , \tag{1}$$

and its the inverse map is written as:

$$\mathrm{L}^{-1}:(x,y)\to(u,v) \text{ such that } u=\sqrt{\frac{y+\sqrt{x^2+y^2}}{2}} \text{ and } v=\frac{y}{\sqrt{2y+2\sqrt{x^2+y^2}}}\ . \tag{2}$$

From (1) and (2), one may easily show some useful relation formula such as

$$\partial_x^2+\partial_y^2=4^{-1}\left(u^2+v^2\right)^{-1}\left(\partial_u^2+\partial_v^2\right) , \tag{3}$$

$$dxdy=4\left(u^2+v^2\right)dudv , \tag{4}$$

$$x\partial_y-y\partial_x=\left(u\partial_v-v\partial_u\right)/2 , \tag{5}$$

which may be applied later.

Now, beginning at the stationary Schrodinger equation describing a two-dimensional particle confining by a Coulomb potential $-Z/r$ combining with a non-Coulomb central potential $\tilde{V}(r)$ in (x,y) space (here it is called "former" problem)[1]

$$(A):\ \left\{-\left(\partial_x^2+\partial_y^2\right)/2-Z/r+\tilde{V}(r)\right\}\Psi(x,y)=E\Psi(x,y), \tag{6}$$

whereas $r=\sqrt{x^2+y^2}$ is the distance from this particle to origin, the transformation (1) allows us to transfer equation (6) into another stationary Schrodinger equation describing a two-dimensional particle moving under a harmonic potential $\omega^2\rho^2/2$ adding a non-squared central potential $\tilde{\mathrm{V}}(\rho)$ in (u,v) space (called "later" one) with $\rho=\sqrt{u^2+v^2}$

$$(B):\ \left\{-\left(\partial_u^2+\partial_v^2\right)/2+\omega^2\rho^2/2+\tilde{\mathrm{V}}(\rho)\right\}\psi(u,v)=\mathrm{E}\psi(u,v) . \tag{7}$$

In detail, the map between these equations (6) and (7)

$$(A)\xrightarrow{\mathrm{L}}(B) , \tag{8}$$

includes relationship between two wave functions:

$$\Psi(x,y)=\psi(u,v), \tag{9}$$

as well as the potentials and energies

$$\mathrm{E}=4Z,\quad E=-\omega^2/8,\quad \mathrm{V}(\rho)=4\rho^2V\left(\rho^2\right). \tag{10}$$

For example, if (A) is Coulomb problem $V(r)=0$, (B) becomes a harmonic oscillator $\mathrm{V}(\rho)=0$ [8]. Another example is that (B) is sextic DWAO problem $\omega^2<0$ and $\mathrm{V}(\rho)=2\Omega^2\rho^6$ when (A) is C+HO problem $V(r)=\Omega^2r^2/2$ (Hoang-Do, 2016; Hoang-Do, Pham & Le, 2013; Le, Hoang & Le, 2017, 2018).

2.2. Family of problems related to Coulomb plus harmonic oscillator (C+HO) and sextic double well anharmonic oscillator (sextic DWAO) problems

Within the scheme described above, the authors apply the map (8), i.e transformation (1), repeatedly to the following already-known connection

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[1] Here the authors use the unit such that $\hbar=1, m=1, e=1$ to avoid the cumbersome from dimension of these problems.

$$\left(C+HO\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(\text{sextic DWAO}\right),$$

whereas their stationary Schrodinger equations are

$$\left(C+HO\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-\frac{a_3}{r}+b_3r^2\right\}\Psi^{(3)}\left(x,y\right)=E_3\Psi^{(3)}\left(x,y\right), \tag{11}$$

$$\left(\text{sextic DWAO}\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-a_4r^2+b_4r^6\right\}\Psi^{(4)}\left(x,y\right)=E_4\Psi^{(4)}\left(x,y\right), \tag{12}$$

to create a series of interrelated problems such as

$$\ldots\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(C+HO\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(\text{sextic DWAO}\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\ldots\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\ldots. \tag{13}$$

Relation (10) between potentials and energies of two interrelated problems shows that energy of former problem E vanishes when harmonic oscillator term of later problem disappears $\omega^2=0$; vice versa, energy of later problem E vanishes if Coulomb term absents in potential of former problem $Z=0$. Meanwhile, the map (8) cannot be applied endlessly and the series (13) must start at the first problem including Coulomb potential with zero energy and end at the first problem containing harmonic term with zero energy. As a result, the series (13) solely contains five members named as follow:

$$\begin{gathered}\left(G1\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(G2\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(G3=C+HO\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\\ \underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(G4=\text{sextic DWAO}\right)\underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}}\left(G5\right)\end{gathered}. \tag{14}$$

From procedure described in Subsection 2.1, the authors find out the Schrodinger equations for each member of above series (14) and show as a list below:

$$\left(G1\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-\frac{a_1}{r^{7/4}}-\frac{b_1}{r^{3/2}}+\frac{c_1}{r}\right\}\Psi^{(1)}\left(x,y\right)=E_1\Psi^{(1)}\left(x,y\right), \tag{15}$$

$$\left(G2\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-\frac{a_2}{r^{3/2}}-\frac{b_2}{r}\right\}\Psi^{(2)}\left(x,y\right)=E_2\Psi^{(2)}\left(x,y\right), \tag{16}$$

$$\left(G3\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-\frac{a_3}{r}+b_3r^2\right\}\Psi^{(3)}\left(x,y\right)=E_3\Psi^{(3)}\left(x,y\right), \tag{17}$$

$$\left(G4\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-a_4r^2+b_4r^6\right\}\Psi^{(4)}\left(x,y\right)=E_4\Psi^{(4)}\left(x,y\right), \tag{18}$$

$$\left(G5\right):\quad \left\{-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)-c_5r^2-a_5r^6+b_5r^{14}\right\}\Psi^{(5)}\left(x,y\right)=E_5\Psi^{(5)}\left(x,y\right), \tag{19}$$

in which parameter b_j of their potentials are positive so that these Schrodinger equations are bounded in $L^2(\mathbb{R}^2)$ Hilbert space. These parameters with energies E_j in equations (14) to (19) are linked together by relation (10):

$$(G1)\underset{L^{-1}}{\overset{L}{\rightleftarrows}}(G2):\begin{cases} a_2 = 4a_1, & b_2 = 4b_1, \\ E_2 = -4c_1, & E_1 = 0, \end{cases} \tag{20}$$

$$(G2)\underset{L^{-1}}{\overset{L}{\rightleftarrows}}(G3):\{a_3 = 4a_2, \quad b_3 = -4E_2, \quad E_3 = 4b_2, \tag{21}$$

$$(G3)\underset{L^{-1}}{\overset{L}{\rightleftarrows}}(G4):\{a_4 = 4E_3, \quad b_4 = 4b_3, \quad E_4 = 4a_3, \tag{22}$$

$$(G4)\underset{L^{-1}}{\overset{L}{\rightleftarrows}}(G5):\begin{cases} c_5 = 4E_4, & a_5 = 4a_4, \\ b_5 = 4b_4, & E_5 = 0. \end{cases} \tag{23}$$

Before going to the next Subsection, the following figure illustrates $V(r)$ potential functions for (G1)-(G5) problems described in equations (15)-(19).

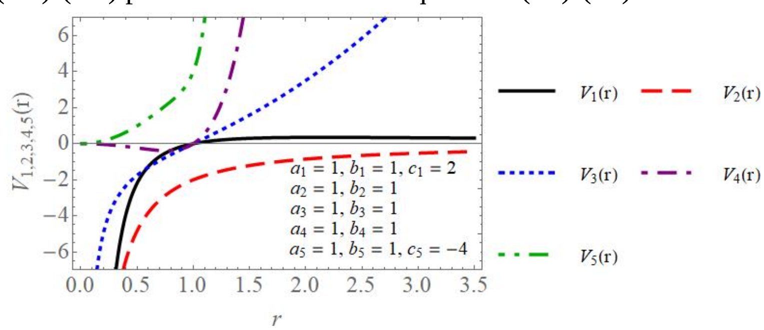

Figure 1. *Potentials for (G1)-(G5) problems at certain values of parameters. The black-thick, red-dashed, blue-dotted, purple-dot-dash-dot and green-dot-dot-dash lines describe potentials of G1, G2, G3, G4 and G5 problems respectively. These potentials support for existence of bound states when b_j parameters are positive ($b_j = 1$).*

2.3. Description about wave function ansatz method based on supersymmetry nature of Schrodinger equations

As mentioned in introduction, among these five problems in the constructed family, the third (G3) and the forth (G4) one, which are actually C+HO and sextic DWAO problems, has been analytically solved by Taut (1995), Le, Hoang & Le (2017, 2018). These exact analytical solutions, which are often classified as quasi-exact solutions, were obtained by their supersymmetry nature (Cooper, Khare & Sukhatme, 2000) and a method named wave function ansatz (Dong, 2011). Hence, wave function ansatz method based on supersymmetry nature of other problems (G1, G2 and G5) is now applied to figure out their quasi-exact solutions.

First of all, the authors present a brief about the procedure using wave function ansatz method based on supersymmetry nature to solve Schrodinger equation of a particle confined by a central potential $V(r)$:

$$\left\{-\left(\partial_x^2+\partial_y^2\right)/2+V(r)\right\}\Psi(x,y)=E\Psi(x,y)\ . \tag{24}$$

Due to the cylinder symmetry corresponding to the integral of motion $\hat{L}_z=-\imath\left(x\partial_y-y\partial_x\right)$, the wave function can be variable-seperated in polar coordinate (r,θ):

$$\Psi_{n,m}(x,y)\sim r^{-1/2}\psi_{n,m}(r)e^{im\theta}, \tag{25}$$

whereas radial wave function $\psi(r)$ is governed by an one-dimension Schrodinger equation

$$\left\{-\frac{1}{2}\partial_r^2+V_{eff}(r)\right\}\psi_{n,m}(r)=E_n\psi_{n,m}(r). \tag{26}$$

describing a one-dimension motion over the domain $[0,+\infty)$ under an effective potential

$$V_{eff}(r)=\left(m^2-1/4\right)/\left(2r^2\right)+V(r). \tag{27}$$

If this effective potential can be written as the following form

$$V_{eff}(r)=W_{eff}^2(r)/2+W_{eff}'(r)/2+E_0, \tag{28}$$

then supersymmetry nature of equation (26) exists and consequently, the zero-node radial wave function $\psi_{0,m}(r)$ must be (Cooper, Khare & Sukhatme, 2000):

$$\psi_{0,m}(r)\sim\exp\left(\int_0^r W_{eff}(r')dr'\right), \tag{29}$$

corresponds to lowest level E_0 . From zero-node radial wave function $\psi_{0,m}(r)$ in (29), the authors can generate the higher state radial wave function $\psi_{n,m}(r)$ by making an ansatz:

$$\psi_{n,m}(r)=\psi_{0,m}(r)\,f_{n,m}(r)\sim\exp\left(\int_0^r W_{eff}(r')dr'\right)\times f_{n,m}(r)\ , \tag{30}$$

in which the function $f_{n,m}(r)$ has n nodes. This ansatz is known as wave function ansatz (Dong, 2011). The simplest form of generator function $f_{n,m}(r)$ to create the nodes for radial wave function is finite polynomial such as:

$$f_{n,m}(r)=\sum_{k=0}^{D}C_k r^{\delta k}\ . \tag{31}$$

Substituting (30) and (31) into equation (26), the authors obtain the recurrence relation between the expansion coefficient C_k:

$$C_{p+k} = g\left(C_p, C_{p+1}, \ldots, C_{p+k-1}\right). \tag{32}$$

Since the function $f_{n,m}(r)$ in equation (31) is a finite polynomial, the series of coefficients C_k must be cut-off at a certain degree D, hence, the following constraint must be satisfied

$$C_D \neq 0, \quad C_{D+1} = C_{D+2} = \ldots = 0. \tag{33}$$

Meanwhile, solely some certain values of potentials' parameters a_j, b_j, c_j are allowed to achieve this class of exact analytical solution. This is the reason why this class is often called quasi-exact solution in some literatures. Besides, energy levels are also obtained from solving the constraint (33). It is noticed that the constraint (33) still hold even for zero-node radial wave function; thus, writing down effective potential as the form of (28) sometimes may requires some conditions.

3. Results

Based on the procedure presented above, the key to obtain quasi-exact solution is to find the superpotential $W_{eff}(r)$ in the formula (28). Table 1 below shows the superpotential $W_{eff}(r)$ for G1, G2, G3, G4 and G5 problems. The constraint allowing to write effective potentials as the form of (28) is also given in Table 1.

Table 1. *Effective potential $V_{eff}(r)$ and superpotential $W_{eff}(r)$ of G1, G2, G3, G4 and G5 problems*

Problem	Effective potential $V_{eff}(r)$	Condition for zero-node solution	Superpotential $W_{eff}(r)$	E_0
G1	$\frac{m^2-1/4}{2r^2} - \frac{a_1}{r^{7/4}} - \frac{b_1}{r^{3/2}} + \frac{c_1}{r}$	$a_1 = 0$ $c_1 = \frac{b_1^2}{2\left(\lvert m\rvert+1/4\right)^2}$	$\frac{\lvert m\rvert+1/2}{r} - \frac{b_1}{\left(\lvert m\rvert+1/4\right)\sqrt{r}}$	0
G2	$\frac{m^2-1/4}{2r^2} - \frac{a_2}{r^{3/2}} - \frac{b_2}{r}$	$a_2 = 0$	$\frac{\lvert m\rvert+1/2}{r} - \frac{b_2}{\lvert m\rvert+1/2}$	$-\frac{b_2^2}{2\left(\lvert m\rvert+1/2\right)^2}$
G3	$\frac{m^2-1/4}{2r^2} - \frac{a_3}{r} + b_3 r^2$	$a_3 = 0$	$\frac{\lvert m\rvert+1/2}{r} - r\sqrt{2b_3}$	$\sqrt{2b_3}\left(\lvert m\rvert+1\right)$
G4	$\frac{m^2-1/4}{2r^2} - a_4 r^2 + b_4 r^6$	$a_4 = \left(\lvert m\rvert+2\right)\sqrt{2b_4}$	$\frac{\lvert m\rvert+1/2}{r} - r^3\sqrt{2b_4}$	0
G5	$\frac{m^2-1/4}{2r^2} - c_5 r^2 - a_5 r^6 + b_5 r^{14}$	$a_5 = \left(\lvert m\rvert+4\right)\sqrt{2b_5}$ $c_5 = 0$	$\frac{\lvert m\rvert+1/2}{r} - r^7\sqrt{2b_5}$	0

From above superpotential, the authors obtain the appropriate wave function ansatz for each problem as given by follow:

$$(G1): \quad \psi_{n,m}^{(1)}(r) \sim r^{|m|+1/2} \exp\left(-2\sqrt{2c_1 r}\right) \times f_{n,m}^{(1)}(r), \tag{34}$$

$$(G2):\ \ \psi^{(2)}_{n,m}(r) \sim r^{|m|+1/2}\exp\left(-\sqrt{-2E_2}\ r\right)\times f^{(2)}_{n,m}(r), \tag{35}$$

$$(G3):\ \ \psi^{(3)}_{n,m}(r) \sim r^{|m|+1/2}\exp\left(-\sqrt{2b_3}r^2/2\right)\times f^{(3)}_{n,m}(r), \tag{36}$$

$$(G4):\ \ \psi^{(4)}_{n,m}(r) \sim r^{|m|+1/2}\exp\left(-\sqrt{2b_4}r^4/4\right)\times f^{(4)}_{n,m}(r), \tag{37}$$

$$(G5):\ \ \psi^{(5)}_{n,m}(r) \sim r^{|m|+1/2}\exp\left(-\sqrt{2b_5}r^8/8\right)\times f^{(5)}_{n,m}(r). \tag{38}$$

According to Le, Hoang & Le (2017, 2018), $f^{(3)}_{n,m}(r)$ and $f^{(4)}_{n,m}(r)$ of (G3) and (G4) problems were written as

$$f^{(3)}_{n,m}(r)=\sum_{j=0}^{D}C_j r^j, \quad \text{and} \quad f^{(4)}_{n,m}(r)=\sum_{j=0}^{D}C_j r^{2j} \tag{39}$$

which correspond to set $\delta=1$ and $\delta=2$ in equation (31) respectively. Hence, the suitable forms for $f^{(1)}_{n,m}(r), f^{(2)}_{n,m}(r), f^{(5)}_{n,m}(r)$ functions must be

$$f^{(1)}_{n,m}(r)=\sum_{k=0}^{D}\frac{2^k\sqrt{2^k}\,(4|m|)!}{k!(k+4|m|)!}C^{(1)}_k r^{k/4}, \tag{40}$$

$$f^{(2)}_{n,m}(r)=\sum_{k=0}^{D}\frac{2^k\,(4|m|)!}{k!(k+4|m|)!}C^{(2)}_k r^{k/2}, \tag{41}$$

$$f^{(5)}_{n,m}(r)=\sum_{k=0}^{D}\frac{\Gamma(1+|m|/2)}{\Gamma(k+1+|m|/2)k!\sqrt{2^k}}C^{(5)}_k r^{4k}, \tag{42}$$

which are equivalent to put $\delta=1/4$, $\delta=1/2$ and $\delta=4$ in equation (31).

Replacing (34), (35) and (38) whereas $f^{(1)}_{n,m}(r), f^{(2)}_{n,m}(r), f^{(5)}_{n,m}(r)$ are defined as (40), (41) and (42) into (G1), (G2) and (G5) problems respectively yields us the recurrence relations of expansion coefficient C_k:

$$(G1): C^{(1)}_{k+2}=\alpha^{(1)}_{k+2}C^{(1)}_{k+1}-\left(\beta^{(1)}_{k+1}\right)^2 C^{(1)}_k \quad \text{when } E_1=0, \tag{43}$$

$$(G2): C^{(2)}_{k+2}=\alpha^{(2)}_{k+2}C^{(2)}_{k+1}-\left(\beta^{(2)}_{k+1}\right)^2 C^{(2)}_k, \tag{44}$$

$$(G5): C^{(5)}_{k+2}=\alpha^{(5)}_{k+2}C^{(5)}_{k+1}-\left(\beta^{(5)}_{k+1}\right)^2 C^{(5)}_k \quad \text{when } E_5=0, \tag{45}$$

which belongs to the class of three recurrence relations whose solution is the well-known the determinant of a tridiagonal matrix (Muir & Metzler, 1960):

$$(G1): C_k^{(1)} = \begin{vmatrix} \alpha_1^{(1)} & \beta_1^{(1)} & & & 0 \\ \beta_1^{(1)} & \alpha_2^{(1)} & \beta_2^{(1)} & & \\ & \beta_2^{(1)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{k-1}^{(1)} \\ 0 & & & \beta_{k-1}^{(1)} & \alpha_k^{(1)} \end{vmatrix}, \tag{46}$$

with $\alpha_i^{(1)} = -8\sqrt{2}a_1, \quad \beta_i^{(1)} = \sqrt{i\left(i+8|m|\right)\left(4b_1 - \left(i+4|m|\right)\sqrt{2c_1}\right)},$

$$(G2): C_k^{(2)} = \begin{vmatrix} \alpha_1^{(2)} & \beta_1^{(2)} & & & 0 \\ \beta_1^{(2)} & \alpha_2^{(2)} & \beta_2^{(2)} & & \\ & \beta_2^{(2)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{k-1}^{(2)} \\ 0 & & & \beta_{k-1}^{(2)} & \alpha_k^{(1)} \end{vmatrix}, \tag{47}$$

with $\alpha_i^{(2)} = -4a_2, \quad \beta_i^{(2)} = \sqrt{i\left(i+4|m|\right)\left(2b_2 - \left(i+2|m|\right)\sqrt{-2E_2}\right)}$

$$(G5): C_k^{(5)} = \begin{vmatrix} \alpha_1^{(5)} & \beta_1^{(5)} & & & 0 \\ \beta_1^{(5)} & \alpha_2^{(5)} & \beta_2^{(5)} & & \\ & \beta_2^{(5)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{k-1}^{(5)} \\ 0 & & & \beta_{k-1}^{(5)} & \alpha_k^{(5)} \end{vmatrix}, \tag{48}$$

with $\alpha_i^{(5)} = -c_5\sqrt{2}$, and $\beta_i^{(5)} = \sqrt{i\left(i+|m|/2\right)\left(a_5/4 - \left(i+|m|/4\right)\sqrt{2b_5}\right)}$

Then, from cut-off condition (33), which is equivalent to the following equation

$$\beta_{D+1}^{(j)} = 0, \quad \begin{vmatrix} \alpha_1^{(j)} & \beta_1^{(j)} & & & 0 \\ \beta_1^{(j)} & \alpha_2^{(j)} & \beta_2^{(j)} & & \\ & \beta_2^{(j)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_D^{(j)} \\ 0 & & & \beta_D^{(j)} & \alpha_{D+1}^{(j)} \end{vmatrix} = 0 , \tag{49}$$

corresponding energy levels and allowing constraints for potentials' parameters can be obtained. Quasi-exact wave functions, corresponding energy levels and allowing constraints for potentials' parameters of G1, G2 and G5 problems are shown in Table 2.

Table 2. *Quasi-exact wave functions, corresponding energy levels and allowing constraints for potentials' parameters of G1, G2 and G5 problems*

	(G1)	(G2)	(G5)
Potential	$-\frac{a_1}{r^{7/4}}-\frac{b_1}{r^{3/2}}+\frac{c_1}{r}$	$-\frac{a_2}{r^{3/2}}-\frac{b_2}{r}$	$-c_5 r^2 - a_5 r^6 + b_5 r^{14}$
Wave function	$r^{\lvert m\rvert+1/2}\exp\left(-\frac{2b_1}{\left(\lvert m\rvert+1/4\right)}\sqrt{r}\right)\times\sum_{k=0}^{D} C_k^{(1)} r^{k/4}\times\frac{e^{im\theta}}{\sqrt{2\pi}}$	$r^{\lvert m\rvert+1/2}\exp\left(-\frac{b_2}{\lvert m\rvert+1/2}r\right)\times\sum_{k=0}^{D} C_k^{(2)} r^{k/2}\times\frac{e^{im\theta}}{\sqrt{2\pi}}$	$r^{\lvert m\rvert+1/2}\exp\left(-\frac{\sqrt{2b_5}\,r^8}{8}\right)\times r^{\mathrm{mod}(m,4)}\sum_{k=0}^{D} C_k^{(5)} r^{4k}\times\frac{e^{im\theta}}{\sqrt{2\pi}}$
Energy levels	$E=0$	$E_2=-\frac{2b_2^2}{\left(D+1+2\lvert m\rvert\right)^2}$	$E=0$
Constraint	$c_1=\frac{b_1^2}{2\left(\left(D+1\right)/4+\lvert m\rvert\right)^2}$ $\begin{vmatrix} \alpha_1^{(1)} & \beta_1^{(1)} & & & 0 \\ \beta_1^{(1)} & \alpha_2^{(1)} & \beta_2^{(1)} & & \\ & \beta_2^{(1)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{D+1}^{(1)} \\ 0 & & & \beta_{D+1}^{(1)} & \alpha_{D+2}^{(1)} \end{vmatrix}=0$	$\begin{vmatrix} \alpha_1^{(5)} & \beta_1^{(5)} & & & 0 \\ \beta_1^{(5)} & \alpha_2^{(5)} & \beta_2^{(5)} & & \\ & \beta_2^{(5)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{D+1}^{(5)} \\ 0 & & & \beta_{D+1}^{(5)} & \alpha_{D+2}^{(5)} \end{vmatrix}=0$	$a_5=\left(4D+4+\lvert m\rvert\right)\sqrt{2b_5}$ $\begin{vmatrix} \alpha_1^{(5)} & \beta_1^{(5)} & & & 0 \\ \beta_1^{(5)} & \alpha_2^{(5)} & \beta_2^{(5)} & & \\ & \beta_2^{(5)} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{D+1}^{(5)} \\ 0 & & & \beta_{D+1}^{(5)} & \alpha_{D+2}^{(5)} \end{vmatrix}=0$

Then some certain allowed parameters and energy levels for G1, G2 and G5 problems are found by solving the constraint equations given in Table 2 via some calculations in Mathematica. For illustrative purpose, Table 3 below shows numerical values of some allowed parameters, energy levels for some bound states in G1, G2 and G5 problems.

Table 3. *Numerical values of some allowed parameters a_j, b_j, c_j and exact energy levels E_j for some bound states (n,m) in G_j problems where j = 1, 2, 5. These values can be used as reference for testing numerical methods*

G1 problem					
D	***a_1***	***b_1***	***c_1***	***(n,m)***	***Energy E_1***
0	0.00000000000000	1.00000000000000	0.09876543209877	(0,-2)	0.00000000000000
	0.00000000000000	1.00000000000000	0.32000000000000	(0,-1)	0.00000000000000
	0.00000000000000	1.00000000000000	8.00000000000000	(0,0)	0.00000000000000
	0.00000000000000	1.00000000000000	0.32000000000000	(0,1)	0.00000000000000
	0.00000000000000	1.00000000000000	0.09876543209877	(0,2)	0.00000000000000
1	-	1.00000000000000	0.08000000000000	(0,-2)	0.00000000000000
	0.23048861143232	1.00000000000000	0.08000000000000	(1,-2)	0.00000000000000
	0.23048861143232	1.00000000000000	0.22222222222222	(0,-1)	0.00000000000000

	- 0.21650635094611 0.21650635094611 - 0.12500000000000 0.12500000000000 - 0.21650635094611 0.21650635094611 - 0.23048861143232 0.23048861143232	1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000	0.22222222222222 2.00000000000000 2.00000000000000 0.22222222222222 0.22222222222222 0.08000000000000 0.08000000000000	(1,-1) (0,0) (1,0) (0,1) (1,1) (0,2) (1,2)	0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000
2	- 0.44594129250792 0.00000000000000 0.44594129250792 - 0.41187723552396 0.00000000000000 0.41187723552396 - 0.25000000000000 0.00000000000000 - 0.25000000000000 - 0.41187723552396 0.00000000000000 0.41187723552396 - 0.44594129250792 0.00000000000000 0.44594129250792	1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000	0.06611570247934 0.06611570247934 0.06611570247934 0.16326530612245 0.16326530612245 0.16326530612245 0.88888888888889 0.88888888888889 0.88888888888889 0.16326530612245 0.16326530612245 0.16326530612245 0.06611570247934 0.06611570247934 0.06611570247934	(0,-2) (1,-2) (2,-2) (0,-1) (1,-1) (2,-1) (0,0) (1,0) (2,0) (0,1) (1,1) (2,1) (0,2) (1,2) (2,2)	0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000

G2 problem

D	***a_2***	***b_2***	***(n,m)***	***Energy E_2***
0	0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000 0.00000000000000	1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000	(0,-2) (0,-1) (0,0) (0,1) (0,2)	- 0.04000000000000 - 0.11111111111111 - 1.00000000000000 - 0.11111111111111 - 0.04000000000000

1	-0.67977010278300 0.67977010278300 -0.54179672970479 0.54179672970479 -0.28426365609385 0.28426365609385 -0.54179672970479 0.54179672970479 -0.67977010278300 0.67977010278300	1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000	(0,-2) (1,-2) (0,-1) (1,-1) (0,0) (1,0) (0,1) (1,1) (0,2) (1,2)	- 0.02777777777778 - 0.02777777777778 - 0.06250000000000 - 0.06250000000000 - 0.25000000000000 - 0.25000000000000 - 0.06250000000000 - 0.06250000000000 - 0.02777777777778 - 0.02777777777778
2	-1.24159589918019 0.00000000000000 1.24159589918019 -1.00563751900533 0.00000000000000 1.00563751900533 -0.59986244844551 0.00000000000000 0.59986244844551 -1.00563751900533 0.00000000000000 1.00563751900533 -1.24159589918019 0.00000000000000 1.24159589918019	1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000 1.00000000000000	(0,-2) (1,-2) (0,-2) (0,-1) (1,-1) (0,-1) (0,0) (1,0) (2,0) (0,1) (1,1) (0,1) (0,2) (1,2) (0,2)	- 0.02040816326531 - 0.02040816326531 - 0.02040816326531 - 0.04000000000000 - 0.04000000000000 - 0.04000000000000 - 0.11111111111111 - 0.11111111111111 - 0.11111111111111 - 0.04000000000000 - 0.04000000000000 - 0.04000000000000 -

				0.02040816326531 - 0.02040816326531 - 0.02040816326531

G5 problem					
D	***a5***	***b5***	***c5***	***(n,m)***	***Energy Ẻ$_5$***
0	8.4852813742386	1.0000000000000	0.0000000000000	(0,-2)	0.0000000000000
	7.0710678118655	1.0000000000000	0.0000000000000	(0,-1)	0.0000000000000
	5.6568542494924	1.0000000000000	0.0000000000000	(0,0)	0.0000000000000
	7.0710678118655	1.0000000000000	0.0000000000000	(0,1)	0.0000000000000
	8.4852813742386	1.0000000000000	0.0000000000000	(0,2)	0.0000000000000
1	14.1421356237310	1.0000000000000	-1.1892071150027	(0,-2)	0.0000000000000
	14.1421356237310	1.0000000000000	1.1892071150027	(1,-2)	0.0000000000000
	12.7279220613579	1.0000000000000	-1.0298835719536	(0,-1)	0.0000000000000
	12.7279220613579	1.0000000000000	1.0298835719536	(1,-1)	0.0000000000000
	11.3137084989848	1.0000000000000	-0.8408964152537	(0,0)	0.0000000000000
	11.3137084989848	1.0000000000000	0.8408964152537	(1,0)	0.0000000000000
	12.7279220613579	1.0000000000000	-1.0298835719536	(0,1)	0.0000000000000
	12.7279220613579	1.0000000000000	1.0298835719536	(1,1)	0.0000000000000
	14.1421356237310	1.0000000000000	-1.1892071150027	(0,2)	0.0000000000000
	14.1421356237310	1.0000000000000	1.1892071150027	(1,2)	0.0000000000000
2	19.7989898732233	1.0000000000000	-2.6591479484725	(0,-2)	0.0000000000000
	19.7989898732233	1.0000000000000	0.0000000000000	(1,-2)	0.0000000000000
	19.7989898732233	1.0000000000000	2.6591479484725	(2,-2)	0.0000000000000
	18.3847763108502	1.0000000000000	-2.3784142300054	(0,-1)	0.0000000000000
	18.3847763108502	1.0000000000000	0.0000000000000	(1,-1)	0.0000000000000
	18.3847763108502	1.0000000000000	2.3784142300054	(2,-1)	0.0000000000000
	16.9705627484771	1.0000000000000	-2.0597671439071	(0,0)	0.0000000000000
	16.9705627484771	1.0000000000000	0.0000000000000	(1,0)	0.0000000000000
	16.9705627484771	1.0000000000000	2.0597671439071	(2,0)	0.0000000000000
	18.3847763108502	1.0000000000000	-2.3784142300054	(0,1)	0.0000000000000
	18.3847763108502	1.0000000000000	0.0000000000000	(1,1)	0.0000000000000
	18.3847763108502	1.0000000000000	2.3784142300054	(2,1)	0.0000000000000
	19.7989898732233	1.0000000000000	-2.6591479484725	(0,2)	0.0000000000000
	19.7989898732233	1.0000000000000	0.0000000000000	(1,2)	0.0000000000000
	19.7989898732233	1.0000000000000	2.6591479484725	(2,2)	0.0000000000000

4. Discussion

Now, the authors perform observation about the number of exact bound states for G5 problem as an example. Since only zero energy level of G5 problem is exactly solved, the degeneracy of this level for each value of (a_5, b_5, c_5) parameters; meanwhile, the authors count how many states (n,m) corresponding to zero energy level for each value of (a_5,b_5,c_5). In order to do this, let us remind the constraint of (a_5,b_5,c_5) for G5 problem to have exact solution at zero energy level:

$$a_5=(4D+4+|m|)\sqrt{2b_5}\ , \tag{50}$$

$$\begin{vmatrix} -c_5 & \tilde{\beta}_1 & & & 0 \\ \tilde{\beta}_1 & -c_5 & \tilde{\beta}_2 & & \\ & \tilde{\beta}_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \tilde{\beta}_D \\ 0 & & & \tilde{\beta}_D & -c_5 \end{vmatrix}=0, \tag{51}$$

Where as $\tilde{\beta}_i=\sqrt[4]{b_5/2}\sqrt{(D+1-i)i(i+|m|/4)}$. Condition (51) means that c_5 is eigenvalue of a Hermitian tridiagonal matrix; hence, according to Sturm sequence theorem, there are $D+1$ different solutions for c_5 (Wilkinson, 1965),

$$c_{5,1}<c_{5,2}<\ldots<c_{5,D+1}. \tag{52}$$

Especially, since both c_5 and $-c_5$ are two opposite solutions of (51), $c_5=0$ is one of the solution of (51) if D is even. In addition, the differential equation of f_5 can be understood as eigensystem problem in which f_5 is eigenfunction and c_5 is eigenvalue; then, Sturm-Liouville theorem states that the node number of eigenfunction $f_5(r)$ corresponding to $D+1$ different solutions of c_5 in (52) must be from 0 to D (Zettl, 2005). As the consequence, the node number of eigenfunction f_5 must be $D/2$ when $c_5=0$ and D is even. Moreover, from condition (50), in total, there are

$$g_5(N)=\sum_{\substack{|m|+4D=N\\ D\,even}}1=\begin{cases}2\,\mathrm{int}(N/8)+2 & N\not\vdots\,8\\ N/4+1 & N\vdots\,8\end{cases}, \tag{53}$$

exact-solution states for each set of (a_5,b_5,c_5) where $a_5=(N+4)\sqrt{2b_5}$ and $c_5=0$. The wave function of these states are equivalent to the wave function of G4 problem when $a_4=(N/2+2)\sqrt{2b_4}$. According to the similar investigation of G4 problem by Le, Hoang & Le (2018), the number of exact-solution energy states in G4 problem is

$$g_4(N)=\begin{cases}2\,\mathrm{int}(N/8)+2 & N\not\vdots\,8, N\vdots\,2\\ N/4+1 & N\vdots\,8\end{cases}=g_5(N). \tag{54}$$

Comparison from (54) and (55) show that number of zero energy state of G5 problem is equal to the one of G4 problem when N is even. However, for G5 problem, there are still exact solutions for odd N while the opposite is true for G4 problem. It means Levi-Civita transformation from G4 problem to G5 problem "creates" more exact solutions.

As mentioned in Introduction, this observation is the consequence of the topological nature of Levi-Civita transformation in the quasi-exact solutions for G1-G5 problems. Coming back to the expression of Levi-Civita transformation which connects the $\mathbb{R}^2:(x,y)$ space to the $\mathbb{R}^2:(u,v)$ space, its inverse map only valid to take form as equation (2) when $u \geq 0$; meanwhile, Levi-Civita transformation actually project the Euclidean space $\mathbb{R}^2:(x,y)$ into a half Euclidean space $\mathbb{R}^+ \times \mathbb{R}:(u,v)$. This property of Levi-Civita transformation bases on its topological nature, particularly, on its relation with 0th Hopf map, i.e real Hopf map: $Z_2 = S^0 \, \mathrm{O} \, S^1 \to S^1$ (Bellucci, 2006). It is easily understandable if Levi-Civita transformation is written in polar coordinates instead of Cartesian coordinates:

$$(\rho,\varphi) \underset{\mathrm{L}^{-1}}{\overset{\mathrm{L}}{\rightleftarrows}} (r,\theta): \; r = \rho^2 \quad \theta = 2\varphi \;. \tag{55}$$

Beside the radius part $r = \rho^2$ which expresses Euler identity, the angle part $\theta = 2\varphi$ means that φ lies solely in a half of 1-sphere $S^1 / Z_2 = [0,\pi]$ whenever θ belongs to $S^1 = [0,2\pi]$ sphere of (x,y) space. Such because of this topological property, when two problems are connected by Levi-Civita transformation $(A) \xrightarrow{\mathrm{L}} (B)$, the relationship between the wave functions $\Psi(r,\theta)$ and $\psi(\rho,\varphi)$ of these two connected problems is not simply equal to each other as the statement in equation (9), especially in angular part of wave function. Solving problem (A) and (B) independently in $\mathbb{R}^2$ space provide us the angular part of wave function must be

$$\frac{e^{im_A\theta}}{\sqrt{2\pi}}, \quad \text{and} \quad \frac{e^{im_B\varphi}}{\sqrt{2\pi}}, \quad \text{with } m_A, m_B \text{ is integer,} \tag{56}$$

respectively. The quantization of angular momentums m_A, m_B arises from the periodic condition of $\mathbb{R}^2$ space or more precisely, of S^1 sphere. However, due to topological nature of Levi-Civita transformation, periodic condition of $S^1 = [0,2\pi]$ sphere in (x,y) space is equivalent to periodic condition of $S^1 / Z_2 = [0,\pi]$ half sphere in (u,v) space which leads to the even integer m_B quantization; consequently, the wave function of m_A state in (x,y) space is related to the wave function of $m_B = 2m_A$ state in (u,v) space. In other word, the wave function of odd m_B in (u,v) space does not relate to any bound states' wave function in (x,y) space. Intuitively, this requirement of angular quantum number relation is involved from the fact that the wave function of (B) problem must be symmetry to $u=0$ plane. Figure 2 below show the connection between two spaces and between two wave function.

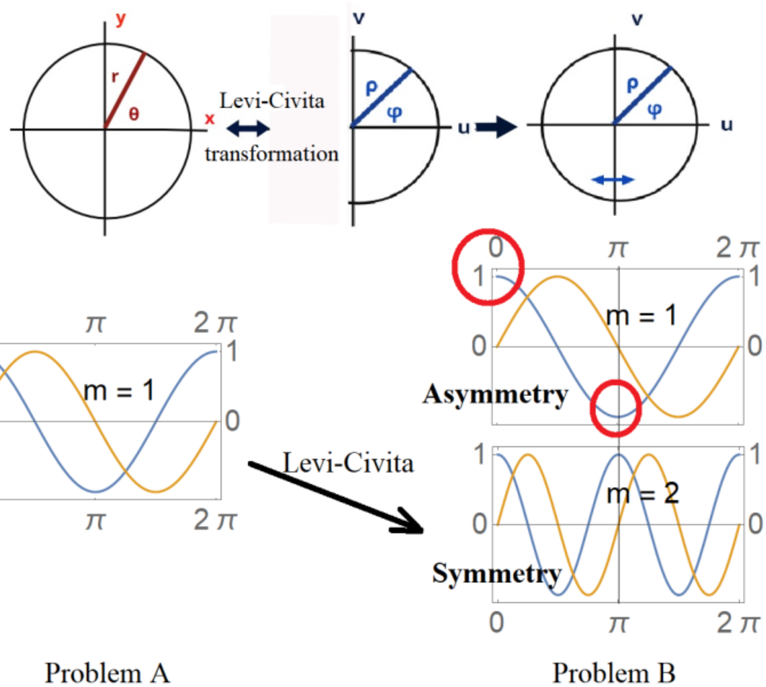

Figure 2. *Relationship between two spaces in Levi-Civita transformation and also between two connected wave function. Only even angular quantum number states of problem B is connected to states of problem A. The reason for this is due to requirement of mirror symmetry at u = 0 plane*

As an example, the following sketch describes angular quantum number of G4 and G5 problems which connects to $m_1=0$ and $m_1=1$ states of G3 problem:

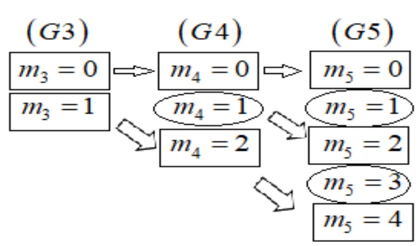

Figure 3. *The sketch describes angular quantum number of G4 and G5 problems connecting to m = 0 and m = 1 states of G3 problem. The rectangles present to states of later problem which connect to states of former problem while ellipses present to "extra" states which do not connect to any state of former problem*

Hence, topological nature of Levi-Civita transformation tell us that the number of bound states for later generation is almost twice as many as the number of bound states for the former. This influence has been observed from the quasi-exact solutions of G4 and G5 problems and can be witnessed even from G1 to G5 problems.

5. Conclusion

In this paper, the authors have constructed a family of problems by repeatedly applying Levi-Civita transformation into C+HO and sextic DWAO problems. Their exact analytical solutions, both energy levels and wave function of bound states, have been obtained and investigated systematically. The relationship between topological property of Levi-Civita transformation and the number of quasi-exact solution in each generation of this family has been discussed also. The results, as mentioned in the introduction, can be good references to test how numerical method is good in solving Schrödinger equation. Furthermore, the next research may be about achieving exact numerical solutions to describe full energy spectrum of each provided problem here by using the scheme of Feranchuk-Komarov operator method.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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MỘT HỌ CÁC PHƯƠNG TRÌNH SCHRÖDINGER KHẢ GIẢI LIÊN HỆ QUA PHÉP BIẾN ĐỔI LEVI-CIVITA

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TÓM TẮT

Một số bài toán hai chiều trong Cơ học lượng tử phi tương đối tính có liên hệ với nhau qua các phép biến đổi không gian nhất định như phép biến đổi Levi-Civita. Tính chất này cho phép xây dựng một họ các bài toán hai chiều có liên hệ mật thiết với nhau. Xuất phát từ hai bài toán có liên hệ với nhau là bài toán Coulomb kết hợp với dao động tử điều hòa và bài toán dao động tử phi điều hòa bậc sáu dạng hố đôi, một họ các bài toán đã được xây dựng bằng cách áp dụng liên tiếp các phép biến đổi Levi-Civita. Một cách hiển nhiên, họ này chứa các bài toán có lời giải giải tích chính xác. Lời giải chuẩn chính xác của từng bài toán chưa biết trong họ này cũng được xây dựng và khảo sát một cách có hệ thống.

Từ khóa: lời giải chuẩn chính xác, tính khả giải giải tích, phép biến đổi Levi-Civita.