TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
TẠP CHÍ KHOA HỌC
KHOA HỌC TỰ NHIÊN VÀ CÔNG NGHỆ
Tập 16, Số 3 (2019): 50-57

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
NATURAL SCIENCES AND TECHNOLOGY
Vol. 16, No. 3 (2019): 50-57

ISSN: 1859-3100

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ON COMINIMAXNESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Received: 04/3/2019; Revised: 15/3/2019; Accepted: 25/3/2019

ABSTRACT

This research introduces and focuses on (I, M)-cominimax modules. The paper shows that if t is an nonnegative integer, M is a finitely generated projective R-module and N is an R-module such that $Ext_R^t(M/IM, N)$ *is minimax and* $H_I^i(N)$ *is (I, M)-cominimax for all* $i < t$, *then* $Hom_R(R/I, H_I^t(M, N))$ *is minimax and* $Ass_R(H_I^t(M, N))$ *is finite.*

Keywords: Generalized local cohomology, (*I*, *M*)-cominimax, minimax modules.

1. Introduction

Throughout this paper, *R* is a commutative noetherian ring and *I* is an ideal of *R*. Let *M* be an *R*-module, the *i*-th local cohomology module of *M* with respect to *I* is denoted by $H_I^i(M)$. Grothendieck, A. (1968) conjectured that if *I* is an ideal of *R* and *N* is a finitely generated *R*-module, then $Hom_R(R/I, H_I^i(M))$ is finitely generated for all $i \geq 0$. Hartshorne, R. (1970) provided a counterexample to this conjecture. He also defined an *R*-module *K* to be *I*-cofinite if $Supp_R(K) \subseteq V(I)$ and $Ext_R^i(R/I, K)$ is finitely generated for all $i \geq 0$, and he asked the following question.

Question. For which rings *R* and ideals *I* are the modules $H_I^i(M)$ *I*-cofinite for all *i* and all finitely generated modules *M* ?

There are some generalizations of the theory of local cohomology. The following one is given by Herzog, J. (1970). Let *j* be a nonnegative integer and *M* a finitely generated *R*-module. Then the *j*-th generalized local cohomology module of *M* and *N* with respect to *I* is defined by

$$H_I^j(M, N) \cong \lim_{\vec{n}}\left(Ext_R^j(M/I^nM, N)\right)$$

If *M* = *R*, then $H_I^j(R, N) = H_I^j(N)$ is the usual local cohomology module. Borna, Sahandi & Yassemi (2011) introduced the concept of (*I*, *M*)-cofinite modules to study the cofiniteness of the module $H_I^j(M, N)$. As an extension of this notion, the researchers define (*I*,*M*)-cominimax modules. An *R*-module *K* is called (*I*, *M*)-cominimax if $Supp_R(K) \subseteq$

$V(I)$ and $Ext_R^i(M/IM, K)$ is minimax for all integer $i \geq 0$. Section 2 shows some properties of these modules in Lemma 2.6, Proposition 2.9 and Theorem 2.10. The last section is devoted to the studying of the minimaxness relating to generalized local cohomology modules. Theorem 3.1 says that if M is a finitely generated projective R-module and $H_I^i(N)$ is (*I, M*)-cominimax for all $i < t$ and $Ext_R^t(M/IM, N)$ is minimax, then $Hom_R(R/I, H_I^t(M, N))$ is minimax, where *t* is a nonnegative integer. It can be seen in Proposition 3.4 that if *M* is a finitely generated projective *R*-module and *N* is an *R*-module such that $H_I^i(M, N)$ is *I*-cominimax for all integer $i \geq 0$, then $Ext_R^i(M/IM, N)$ is minimax for all $i \geq 0$.

2. On (*I, M*)-cominimax modules

Zöschinger, H. (1986) introduced the class of minimax modules. An *R*-module *K* is said to be minimax if there is a finitely generated submodule *T* of *K* such that *K/T* is Artinian.

Remark 2.1.

The following statements hold:

(i) The class of minimax modules contains all finitely generated and all Artinian modules.

(ii) Let $0 \to L \to M \to N \to 0$ be an exact sequence of *R*-modules. Then *M* is minimax if and only if *L* and *N* are both minimax. Thus, any submodule and quotient of a minimax module is minimax. The finite direct sum of minimax modules is minimax. Moreover, if *M* and *N* are two *R*-modules such that *N* is finitely generated and *M* is minimax, then $Ext_R^j(N, M)$ and $Tor_j^R(N, M)$ are minimax for all $j \geq 0$.

(iii) The set of associated primes of any minimax *R*-module is finite.

As a generalization of *I*-cofinite modules, Azami, Naghipour & Vakili (2009) defined the *I*-cominimax modules.

Definition 2.2. (Azami, Naghipour & Vakili, 2009)

An *R*-module *K* is called *I*-cominimax if $Supp_R(K) \subseteq V(I)$ and $Ext_R^i(R/I, K)$ is minimax for all $i \geq 0$.

Note that all *I*-cofinite modules and minimax modules are *I*-cominimax modules. Another extension of *I*-cofinite modules are (*I, M*)-cofinite modules which were introduced by Borna, Sahandi & Yassemi (2011).

Definition 2.3. (Borna, Sahandi & Yassemi, 2011)

Let *M* be an *R*-module. An *R*-module *K* is called (*I, M*)-cofinite if $Supp_R(K) \subseteq V(I)$ and $Ext_R^i(M/IM, K)$ is finitely generated for all $i \geq 0$.

Some properties of (*I, M*)-cofinite modules were shown by Borna, Sahandi & Yassemi (2011). The paper also contained some results on the cofiniteness and the minimaxness concerning generalized local cohomology modules. In a natural way, a new concept that is based on above notions will be given.

Definition 2.4.

Let M be an R-module. An R-module K is (I, M)-cominimax if $Supp_R(K) \subseteq V(I)$ and $Ext_R^i(M/IM, K)$ is minimax for all $i \geq 0$.

The first property shows a relationship between the I-cominimaxness and (I,M)-cominimaxness in the case where M is finitely generated.

Proposition 2.5.

If M is a finitely generated R-module, then I-cominimax modules are (I, M)*-cominimax modules.*

Proof. Assume that K is an I-cominimax R-module. By the hypothesis, $Supp_R(K) \subseteq V(I)$ and $Ext_R^i(R/I, K)$ is minimax for all integer $i \geq 0$. Let $\bar{M} = M/IM$, it is clear that $\bar{M}$ is finitely generated and $Supp_R\bar{M} \subseteq V(I)$. The researchers will show that $Ext_R^i(M/IM, K)$ is minimax by induction on i. By Gruson's theorem (Vasconcelos, 1974, Theorem 4.1), $\bar{M}$ has a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = \bar{M}$$

such that M_j/M_{j-1} is a homomorphic image of $(R/I)^m$ for some positive integer m. Consider short exact sequences

$$0 \to M_{j-1} \to M_j \to M_j/M_{j-1} \to 0$$

where $1 \leq j \leq k$.

Note that there is an isomorphism $Hom_R((R/I)^m, K) \cong Hom_R(R/I, K)^m$. Since K is an I-cominimax R-module, it follows from Remark 2.1.(ii) that $Hom_R((R/I)^m, K)$ is minimax. The exact sequence

$$0 \to Hom_R(M_j/M_{j-1}, K) \to Hom_R(M_j, K) \to Hom_R(M_{j-1}, K) \to \cdots$$

yields that $Hom_R(M_j, K)$ is minimax for all j and then $Hom_R(\bar{M}, K)$ is minimax. Thus we have the conclusion when $i = 0$.

Let $i > 0$. The short exact sequence

$$0 \to M_{j-1} \to M_j \to (R/I)^m \to 0$$

induces a long exact sequence

$$\ldots \to Ext_R^{i-1}(M_{j-1}, K) \to Ext_R^i((R/I)^m, K) \to Ext_R^i(M_j, K) \to \ldots$$

Since $Ext_R^i((R/I)^m, K) \cong Ext_R^i(R/I, K)^m$, it follows that $Ext_R^i(M_1, K)$ is minimax. By inductive hypothesis, $Ext_R^{i-1}(M_{j-1}, K)$ is minimax for all $j \leq k$. By the similar argument as in the proof above, we have $Ext_R^i(M_k, K)$ is minimax and the assertion follows.

Lemma 2.6.

Let $0 \to A \to B \to C \to 0$ *be a short exact sequence of R-modules. If two of three modules are* (*I, M*)*-cominimax, then so is the third.*

Proof. We have that

$Supp_R B = Supp_R A \cup Supp_R C.$

The short exact sequence $0 \to A \to B \to C \to 0$ induces the following exact sequence

$\cdots \to Ext_R^i(M/IM, A) \to Ext_R^i(M/IM, B) \to Ext_R^i(M/IM, C) \to \cdots$

Combining the Remark 2.1.(ii) and the assumption, the claim follows.

Corollary 2.7.

Let $f : A \to B$ *be a homomorphism of* (*I, M*)*-cominimax R-modules. If one of* Ker *f,* Im*f and* Coker*f is* (*I, M*)*-cominimax, then all three are* (*I, M*)*-cominimax.*

It is well-known that in a local ring (*R*, m) if *T* is a finitely generated *R*-module with $Supp_R T \subseteq \{\mathrm{m}\}$, then *T* is an Artinian *R*-module. Now the researchers consider this property in the case where *T* is minimax.

Lemma 2.8.

Let (*R*, m) *be a local ring and T a minimax R-module such that* $Supp_R T \subseteq \{\mathrm{m}\}$. *Then T is Artinian*.

Proof. Assume that *K* is a finitely generated *R*-submodule of *T* such that *T/K* is Artinian. By the hypothesis, $Supp_R K \subseteq Supp_R T \subseteq \{\mathrm{m}\}$, and then *K* is Artinian. Therefore *T* is also an Artinian *R*-module.

Proposition 2.9.

Let (*R*, m) *be a local ring and N is an* (m*, M*)*-cominimax. Then* $Hom_R(M, N)$ *is Artinian.*

Proof. By the hypothesis, $Hom_R(M/\mathrm{m}M, N)$ is minimax. Moreover, there is an isomorphism

$Hom_R(M/\mathrm{m}M, N) \cong Hom_R(R/\mathrm{m}, Hom_R(M, N)).$

It follows from Lemma 2.8 that $Hom_R(R/\mathrm{m}, Hom_R(M, N))$ is an Artinian *R*-module. Moreover,

$Supp_R Hom_R(M, N) \subseteq Supp_R N \subseteq \{\mathrm{m}\}$

which shows that $Hom_R(M, N)$ is m-torsion. By (Melkersson, 1990, Theorem 1.3), the module $Hom_R(M, N)$ is Artinian.

Theorem 2.10.

Let N be an (*I, M*)*-cominimax R-module. Then* $Ass_R(Hom_R(M, N))$ *is finite.*

Proof. The isomorphism

$Hom_R(M/IM, N) \cong Hom_R(R/I, Hom_R(M, N))$

and the hypothesis show that $Hom_R(R/I, Hom_R(M, N))$ is minimax. By Remark 2.1.(iii), the set $Ass_R(Hom_R(R/I, Hom_R(M, N)))$ is finite. On other hand, we have

$Ass_R(Hom_R(R/I, Hom_R(M,N))) = V(I) \cap Ass_R(Hom_R(M,N))$.

Since $Supp_R(N) \subseteq V(I)$ and $Supp_R\big(Hom_R(M,N)\big) \subseteq Supp_R(N)$, it follows that $Ass_R(Hom_R(M,N))$ is finite.

3. On minimaxness

Theorem 3.1.

Let t be a nonnegative integer. Let M be a finitely generated R-module and N an R-module such that $H_I^i(N)$ is (I,M)-cominimax for all integer $i < t$. The following statements hold

(i) $Ext_R^i(M/IM, N)$ *is minimax for all* $i < t$;

(ii) *Assume that M is projective and* $Ext_R^t(M/IM, N)$ *is minimax. Then* $Hom_R(R/I, H_I^t(M,N))$ *is minimax.*

Proof. (i) Let $F = Hom_R(M/IM, -)$ and $G = \Gamma_I(-)$ be functors from the category of *R*-modules to itself. Let *N* be an *R*-module, we see that

$FG(N) = Hom_R(M/IM, \Gamma_I(N)) \cong Hom_R(M/IM, N)$.

If E is an injective R-module, then $G(E) = \Gamma_I(E)$ is also injective by (Brodmann & Sharp, 2013, Proposition 2.1.4). It follows that $R^iF\big(G(E)\big) = 0$. By (Rotman, 2009, Theorem 10.47), there is a Grothendieck spectral sequence

$$E_2^{p,q} = Ext_R^p(M/IM, H_I^q(N))) \underset{p}{\Rightarrow} Ext_R^{p+q}(M/IM, N).$$

Let $n < t$, there is a filtration Φ of submodules of $H^n = Ext_R^n(M/IM, N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $0 \le i \le n$. By the assumption, $E_2^{p,q}$ is minimax for all integers $p \ge 0, q < t$. Since $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$, it follows that $E_\infty^{i,n-i}$ is minimax for all $0 \le i \le n$. Therefore, $\Phi^0 H^n = Ext_R^n(M/IM, N)$ is minimax.

(ii) The proof is by induction on *t*. Let $t = 0$, we have

$$Hom_R(R/I, H_I^0(M,N)) \cong Hom_R(R/I, Hom_R(M,N)) \cong Hom_R(M/IM, N),$$

and the claim follows by the hypothesis.

Let $t > 0$ and $\overline{N} = N/\Gamma_I(N)$. Note that $\overline{N}$ is an *I*-torsion-free *R*-module. By (Brodmann & Sharp, 2013, Corollary 2.1.7), $H_I^i(N) \cong H_I^i(\overline{N})$ for all $i > 0$. The assumption shows that $H_I^i(\overline{N})$ is (*I*,*M*)-cominimax for all $i < t$. The short exact sequence

$$0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$$

gives rise to a long exact sequence

$$\cdots \to Ext_R^i(M/IM, \Gamma_I(N)) \to Ext_R^i(M/IM, N) \to Ext_R^i(M/IM, \overline{N}) \to \cdots$$

Since $\Gamma_I(N)$ is (*I*,*M*)-cominimax and $Ext_R^t(M/IM,N)$ is minimax, it follows that $Ext_R^t(M/IM,\overline{N})$ is minimax. Let $E = E(\overline{N})$ be an injective hull of $\overline{N}$. The short exact sequence

$$0 \to \overline{N} \to E \to E/\overline{N} \to 0$$

induces

$$H_I^i(E/\overline{N}) \cong H_I^{i+1}(N)$$

for all $i \geq 0$ and

$$Ext_R^{t-1}(M/IM,E/\overline{N}) \cong Ext_R^t(M/IM,N).$$

These isomorphisms show that $H_I^i(E/\overline{N})$ is (*I*,*M*)-cominimax for all $i < t-1$ and $Ext_R^{t-1}(M/IM,E/\overline{N})$ is minimax. By the inductive hypothesis, $Hom_R(R/I,H_I^{t-1}(E/\overline{N}))$ is minimax. Moreover, the above short exact sequence also yields the following isomorphisms

$$H_I^i(M,E/\overline{N}) \cong H_I^{i+1}(M,N)$$

for all $i \geq 0$. Consequently, $Hom_R(R/I,H_I^t(M,N))$ is minimax. From the short exact sequence

$$0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$$

there is a long exact sequence

$$\cdots \to H_I^i\big(M,\Gamma_I(N)\big) \to H_I^i(M,N) \to H_I^i(M,\overline{N}) \to \cdots$$

By (Yassemi, Khatami & Sharif, 2002, Lemma 1.1), $H_I^i\big(M,\Gamma_I(N)\big) \cong Ext_R^i(M,\Gamma_I(N))$ for all $i \geq 0$. Since M is a projective R-module, it follows that $H_I^i\big(M,\Gamma_I(N)\big) = 0$ for all $i > 0$. Therefore, $H_I^i(M,N) \cong H_I^i(M,\overline{N})$ and then $Hom_R(R/I,H_I^t(M,N))$ is minimax.

Corollary 3.2.

Let t be a nonnegative integer. Let M be a finitely generated projective R-module and N a minimax R-module such that $H_I^i(N)$ is (I,M)-cominimax for all $i < t$. Then $Hom_R(R/I,H_I^t(M,N))$ is minimax and $Ass_R(H_I^t(M,N))$ is finite.

Proof. By Theorem 3.1 and Remark 2.1.(iii), the set $Ass_R(Hom_R(R/I,H_I^t(M,N)))$ is finite. Moreover,

$$Ass_R(Hom_R(R/I,H_I^t(M,N))) = V(I) \cap Ass_R(H_I^t(M,N)) = Ass_R(H_I^t(M,N)),$$

and the assertion follows.

Corollary 3.3.

Let N be a minimax R-module and t a nonnegative integer such that $H_I^i(N)$ is I-cominimax for all $i < t$. Then $Hom_R(R/I,H_I^t(N))$ is minimax.

Proposition 3.4.

Let M be a finitely generated projective R-module and N an R-module. If $H_I^i(M,N)$ is I-cominimax for all i, then $Ext_R^i(M/IM,N)$ is minimax for all i.

Proof. We now proceed by induction on *i*. When $i = 0$, we have

$Hom_R(M/IM, N) \cong Hom_R(R/I, Hom_R(M, N)) \cong Hom_R(R/I, H_I^0(M, N))$.

Hence $Hom_R(M/IM, N)$ is minimax as $H_I^0(M, N)$ is *I*-cominimax. Let $i > 0$. The short exact sequence

$0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$

induces a long exact sequence

$\cdots \to Ext_R^i(M/IM, \Gamma_I(N)) \to Ext_R^i(M/IM, N) \to Ext_R^i(M/IM, \overline{N}) \to \cdots$

Since *M* is a projective *R*-module, it follows that

$Ext_R^j(M/IM, \Gamma_I(N)) \cong Ext_R^j(R/I, Hom_R(M, \Gamma_I(N))) \cong Ext_R^j(R/I, \Gamma_I(M, N))$

is minimax for all $j \geq 0$. Thus, it is sufficient to prove $Ext_R^j(M/IM, \overline{N})$ is minimax. Let $E = E(\overline{N})$ be the injective envelope of $\overline{N}$, the short exact sequence

$0 \to \overline{N} \to E \to E/\overline{N} \to 0$

induces the isomorphisms

$Ext_R^j(M/IM, E/\overline{N}) \cong Ext_R^{j+1}(M/IM, \overline{N})$

and

$H_I^j(M, E/\overline{N}) \cong H_I^{j+1}(M, \overline{N})$

for all $j \geq 0$. By the assumption and the above argument, $H_I^i(M, \overline{N})$ is *I*-cominimax for all *i*. Hence $H_I^i(M, E/\overline{N})$ is *I*-cominimax for all *i*. It follows from the inductive hypothesis that $Ext_R^{i-1}(M/IM, E/\overline{N})$ is minimax and then $Ext_R^i(M/IM, \overline{N})$ is minimax. This completes the proof.

Corollary 3.5.

Let N be an R-module such that $H_I^i(N)$ is I-cominimax for all i. Then $Ext_R^i(R/I, N)$ is minimax for all i.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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TÍNH COMINIMAX CỦA MÔĐUN ĐỐI ĐỒNG ĐIỀU ĐỊA PHƯƠNG SUY RỘNG

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Ngày nhận bài: 04-3-2019; ngày nhận bài sửa: 15-3-2019; ngày duyệt đăng: 25-3-2019

TÓM TẮT

Chúng tôi giới thiệu và nghiên cứu các môđun (I,M)-cominimax. Bài báo chứng minh rằng nếu t là một số nguyên không âm, M là một R-môđun xạ ảnh hữu hạn sinh và N là một R-môđun thỏa $Ext_R^t(M/IM,N)$ *là minimax và* $H_I^i(N)$ *là (I, M)-cominimax với mọi* $i < t$ *thì* $Hom_R(R/I, H_I^t(M,N))$ *là minimax và* $Ass_R(H_I^t(M,N))$ *là một tập hữu hạn.*

Từ khóa: đối đồng điều địa phương suy rộng, *(I,M)*-minimax, môđun minimax.