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Research Article

DE-MODELING NUMBERS, OPERATIONS AND EQUATIONS: FROM INSIDE-INSIDE TO OUTSIDE-INSIDE UNDERSTANDING

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ABSTRACT

Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

Keywords: number; operation; equation; numeracy; proportionality; early childhood

1. Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish Centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA results caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)". In Germany, the corresponding number is one of five students, according to a plenary address at the Educating Educators conference in Freiburg in October 2019.

This raises some questions: Is mathematics so hard that one out of four or five students cannot master even basic numeracy? Is it mathematics we teach? Do we use the proper psychological learning theories? Can we design a different mathematics curriculum where most students become successful learners? In short: could this be different?

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2. Materials/ Subjects and Methods

To get an answer we use difference research (Tarp, 2018) to create a design research cycle (Bakker, 2018) consisting of reflection, design, and implementation.

2.1. Reflections on Different forms of Mathematics

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-math' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a 'setcentric' (Derrida, 1991), 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant self-reference defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. To avoid self-reference, Russell developed a type theory defining concepts from examples at the abstraction level below. This implies that fractions cannot be numbers, but operators needing numbers to become numbers. Wanting fractions to be rational numbers, the setcentric mathematics neglected Russell's paradox and insisted that to be well defined, a concept must be derived from the mother-concept set above.

In this way, the concept Set changed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism', true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2 x 3 = 6' stating that 2 3s can always be re-counted as 6 1s (Tarp, 2018).

Rejecting setcentrism as making mathematics to abstract, many Anglo-Saxon countries went back to basics and now teach what they call ‘school mathematics’ although this still is mathe-matism adding numbers and fractions without units.

So, today we have three different forms of mathematics: a pre setcentric mathe-matism saying that a function is a calculation with specified and unspecified numbers; a present setcentric meta-matism saying that a function is a subset of a set-product where first-component identity implies second-component identity; and a post setcentric many-math (Tarp, 2018) saying that a function is a number-language sentence with a subject, a verb and a predicate as in the word-language; and that a communicative turn learning language through communication instead of through grammar is needed in the number-language also (Widdowson, 1978).

In its pre and present setcentric versions, a mathematics curriculum typically begins with digits together with addition, later to be followed by subtraction as reversed addition, multiplication as repeated addition, and division as reversed multiplication - sometimes as repeated subtraction also. Then follows fractions, percentages and decimals as rational numbers. Then comes negative numbers, to be followed by expressions with unspecified letter numbers, and by solving equations.

Present setcentric meta-matics defines numbers by inside abstract self-reference as examples of sets. Zero is defined as the empty set. One is defined as the set containing the empty set as its only element. The next numbers then are generated by a follower principle.

With natural numbers defined, integers are defined as equivalence classes in the set-product of natural numbers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross addition holds, $a+d = b+c$. This makes (-2,0) equivalent to (0,2) thus geometrically forming straight lines with gradient 1 in a coordinate system.

With integers defined, rational numbers are defined as equivalence classes the set-product of integers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross multiplication holds, $a \text{ x } d = b \text{ x } c$, thus making (2,3) equivalent to (8,12) thus geometrically forming straight lines with various gradients in a coordinate system.

Equations are examples of open statements that may be transformed into a solution by using abstract algebra’s group theory to neutralize numbers by their inverse numbers.

In geometry, halfplanes define lines that are parallel if a subset relation exists among their halfplanes. And an angle is the intersection set of two halfplanes.

Post setcentric many-math is grounded in the observation that when asked “How old next time?”, a 3year-old will answer “4”, but will object to 4 fingers held together 2 by 2: “That is not 4; that is 2 2s.” So, when adapting to the outside fact Many children count in bundles, and use double-numbers to describe both the numbers of bundles and the bundle-unit. And it turns out that double-numbers contain the core of mathematics since recounting to change units implies proportionality and equations; and when adding double-numbers, on-top addition leads to proportionality making the units like, and next-to addition means adding areas, which leads to integral calculus.

2.2. Reflections on Different forms of Psychology

As institutionalized learning, education is meant to help human brains adapt to the outside world by accommodating schemas failing to assimilate it (Piaget, 1970); or to mediate institutionalized schemas that may colonize the brain (Habermas, 1981).

Adaption is theorized by psychology, often seen as the science of behavior and mind, thus being a sub-discipline of life science, where biology sees life as communities of green and grey cells, plants and animals. Plants stay and get the energy directly from the sun. Animals move to get the energy from plants or other animals, thus needing holes in the head for food and information, making the brain transform stimuli to behavior responses.

Besides the reptile and mammal brains for routines and feelings, humans also have a third human brain for balancing and for storing and sharing information, made possible by transforming forelegs to arms with hands that can grasp (and share) food and things that accompanied by sounds develop a language about the six core outside components: I, you, he-she-it, we, you, and they; or in German: ich, du, er-sie-es, wir, ihr, sie.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, which may be unstructured or structured e.g. by a social institution called education.

Educational psychology first focused on behavior by studying stimulus-response pairings, called classical conditioning where Pavlov showed how dogs would salivate when hearing a sound previously linked to food. Later Skinner (1953) developed this into operant conditioning by adding the concepts of reinforcement and punishment as stimuli following a student response coming from building routines through repetition.

But, does correct responses imply understanding? So, the educational psychology called constructivism focuses on what happens in the mind when constructing inside meaning to outside stimuli. Here especially Piaget (1970), Vygotsky (1986), and Bruner (1977) have contributed in creating teaching methods and practices.

Piaget found four different development stages for children: the sensorimotor stage below 2 years old, the preoperational state from 2 to 7 years old, the concrete operational stage from 7 to 10 years old, and the formal operational stage from 11 years old and up.

In philosophy, existentialism sees existence as preceding essence (Sartre, 2007). Where Piaget sees learning taking place through adaption to outside existence, Vygotsky focuses on adaption to inside institutionalized essence, i.e. through enculturation allowing learners to expand their ‘Zone of Proximal Development’ (ZPD) under the guidance of a more knowledgeable other. “What a child can do today with assistance, she will be able to do by herself tomorrow” (azquotes.com).

Likewise pointing to the importance of good teaching, Bruner developed the concept of instructional scaffolding providing a ladder leading from the ZPD up to a school subject. This should be structured as its university version to help the teacher structure the subject in a way that would give the meaning that the students need for understanding.

Holding that no children master logical thinking before 11 years, and therefore needing to be taught using concrete objects and examples, Piaget instead warned against too much teaching by saying: “Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life” (azquotes.com).

2.3. Merging Mathematics and Psychology

Behaviorism is the educational psychology of pre setcentric mathematics. Present setcentric mathematics instead uses Vygotskian constructivism offering scaffolding from the learners ZPD to the institutionalized setcentric university mathematics as defined by e.g. Freudenthal (1973). However, by its self-referring setcentrism, concepts are no longer defined from examples and counterexamples, but as examples themselves of the more abstract set concept. So now not both rules, procedures, and concepts should be understood. Freudenthal, therefore, recommended a special conference be created called PME, Psychology of Mathematics Education, focusing on how to understand mathematics as described by Skemp (1971) saying “The first part of the book will be concerned with this most basic problem: what *is* understanding, and by what means can we help to bring it about? (p. 14)”. Skemp then uses 123 pages to give an understanding of understanding, even if the inherent self-reference should make one skeptical towards such an endeavor.

Heidegger more directly points to four options when defining something by an is-statement: ‘is for example’ points down to examples and counter-examples, ‘is an example of’ points up to an abstraction, ‘is like’ points over to a metaphor, and ‘is.’ describes existence as something to experience without predicates.

Skemp understands numbers as equivalence cardinality classes in the set of sets being equivalent if connected by bijections. Consequently, children should begin drawing arrows between sets to see if they have the same cardinality that then can be named.

However, this approach met resistance in the classroom as illustrated by in this story:

Teacher: “Here is a set of hats and a set of heads. Is one bigger than the other?” Student: “There are more heads”. Teacher: “Why?” Student: “There are six heads and only five hats." Teacher: "Can you please draw arrows from the hats to the heads!" Student: "No, then one person will not get a hat, and that is unfair."

In his book 'Why Johnny Can't Add', Morris Kline describes other examples of classroom resistance to the New Math, finally rejected by North America, choosing to go 'Back to Basics' even if this meant going back to mathe-matism.

Educational psychology thus has various schools. As an alternative, we might use the observation that children's initial language consists of words that are exemplified in the outside world, thus using personal names instead of pronominals as I and you, and protesting when grandma is named 'Ann'.

Observing that brains easily take in concepts naming outside examples allows formulating a research question: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

2.4. De-modelling Digits

Looking at a modern watch in front of an old building, we realize that Roman numbers and modern Hindu-Arabic numbers are different ways of describing Many.

The Romans used four icons to describe four, or they used one stroke to the left of the letter V iconizing a full hand. Modern numbers use one icon only when rearranging the four sticks or strokes into one 4-icon, which then serves as a unit when counting a total in fours as e.g. T = 3 4s.

We might even say that all digits from zero to nine are icons with as many sticks or strokes as they represent if written less sloppy, where the zero-digit iconizes a magnifying glass finding nothing.

The Romans bundled in 5, 10, 50, 100, 500 and 1000. Modern numbers bundle in tens only, which is written as 10 meaning 1 Bundle and none. However, in education, we may want to symbolize ten with the letter B for 'Bundle'.

2.4.1. Designing and Implementing a micro-curriculum

Based upon the above reflections we now design and implement a micro-curriculum having as its goal to de-model and reify digits as icons. As means we ask the learners to rearrange four sticks in different connecting forms, then five sticks, then six sticks. This is followed by rearranging also other things in icons including themselves, and by walking the icons, etc. Then the learners build routine by exercising writing all digits as icons. As an end product, the learner should be able to rearrange a collection of things in an icon and write down a report using a full number-language sentence with a subject, a verb and a predicate, e.g. "T = 5"; and writing T = B, B1, B2, etc. for ten, eleven, twelve, etc.

2.5. Reflections on how to De-model Bundle-counting Sequences

From early childhood, children memorize the inside sequence of number names ‘one, two…, ten, eleven, twelve, three-ten, four-ten’ etc. Later they learn the symbols corresponding to the different number-names. In some languages, they are lucky to word ‘eleven, twelve, thirteen’ as one-ten, two-ten, three-ten’ etc. In English, number rationality begins with three-ten, making whole populations wonder what eleven and twelve means.

History shows that as most basic English words also these are ‘Anglish’ coming from the Danes settling in England long before the Romans arrived. Thus, with Danish you hear that eleven and twelve means ‘one-left’ and ‘two-left’ coming from Viking counting: ‘eight, nine, ten, 1-left, 2-left, 3-ten’; and ‘1-twotens’ where English shift to ‘twenty-1’.

Likewise, many children and adults wonder why ten has no icon since it has its name as the rest of the digits. Only a few realize that when counting by bundling in tens, ten becomes 1 bundle, or 1B0, or 10 if leaving out the bundle when writing it; even if ten is included when saying it, as e.g. in 63 = sixty-three = 6ten3 = 6B3.

So, it may be an idea to practice different counting sequences that include the name ‘bundle’ so that ‘ten, eleven and twelve’ become ‘1 bundle none, 1 bundle 1, 1 bundle 2’. And it may also be an idea to also count in fives as did the Romans and several East Asian cultures as shown by Chinese and Japanese abacuses. So, we design a lesson about counting fingers first in 5s, then in tens, and later in 4s, 3s, and 2s or pairs.

2.5.1. Designing and Implementing a micro-curriculum

One hand-counted in 5s using B for Bundle: First 1, 2, 3, 4, 5 or B or 1B1; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B.

Two hands counted in 5s: First 1, 2, 3, 4, 5 or B or 1B0, 1B1, …, 1B4, 1B5 or 2B or 2B0; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0, etc.; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B0, 2B-4, …, 2B-1, 2B or 2B0.

Two hands counted in tens: First 1, 2, 3, 4, 5 or half Bundle, 6, 7, 8, 9, ten or full Bundle or 1B0; then 0B1, 0B2,…, 0B9, 0B10 or B or 1B0; then 1B-9, 1B-8, …, 1B-1, 1B.

Two hands counted in 4s is similar to counting in 5s.

Two hands counted in 3s provides the end result T = ten = 3B1 3s. But 3 bundles, 3B, is also 1 bundle of bundles, making 9 = 1BB 3s. So we can also write:

T = ten = 3B1 3s = 1BB1 3s, or T = 1BB0B1 3s, or T = 101 3s.

Two hands counted in 2s provides the end result T = ten = 5B0 2s. But, 2 bundles, 2B, is also 1 bundle of bundles, making 4 = 1 BB 2s; and 2 bundles of bundles, 2BB, is also 1 bundle of bundles of bundles, making 8 = 1 BBB 2s. So we can also write:

T = ten = 5B0 2s = 1BBB 1B 2s = 1BBB 0BB 1B 0 2s = 1010 2s.

This can be illustrated with Lego bricks having different colors where a green 1x2 brick is B, a blue 2x2 brick is BB and a red 4x2 brick is BBB.

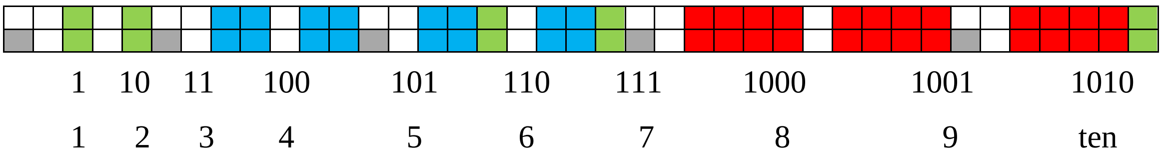


Figure 1. *Ten fingers counted by bundling in 2s and shown by Lego bricks*

2.6. Reflections on how to De-model Operations

Counting a total of eight ones in 2s, we push away 2s using e.g. a playing card that may be iconized as a sloping stroke named division. So, the outside action ‘from 8, push away 2s’ may inside be iconized as ‘8/2’ that gives an inside prediction of what will happen outside:

8/2 times we can perform the action ‘from 8 push away 2’, or T = 8 = 8/2 2s = 4 2s.

Once pushed away, the bundles of 2s may be lifted into a stack of 4 2s. An outside lifting process may be iconized inside by a wooden scissor lifting things when compressed and named multiplication. So, the outside action ‘4 times lifting 2s into a stack’ may inside be iconized as ‘T = 4 x 2 = 4 2s’. And, the reverse outside process ‘de-stack 4 2s into ones’ may inside be predicted by a multiplication T = 4 2s = 4 x 2 = 8.

The total outside process ‘from 8 push away 2s to be stacked as 8/2 2s’ then may be iconized inside as ‘8 = (8/2) x 2’, or ‘*T* = *(T/B)* x *B*’ if we use *T* for the total 8, and *B* for the bundle-unit. By changing units, this ‘bundle-count’ or ‘re-count to change unit’ formula is perhaps the most fundamental formula in mathematics and science, also called the proportionality or linearity formula.

2.6.1. Designing and Implementing a micro-curriculum

Outside action	Inside prediction
From 8, 8/2 times 2s can be pushed away.	8/2 = 4
So, 8 can be recounted in 2s as 8/2 2s	8 = (8/2) x 2
And, *T* can be recounted in Bs as *T/B* Bs	*T* = *(T/B)* x *B*

Figure 2. *Counting 8 in 2s is an example of the recount formula T = (T/B) x B*

Asking “How many 2s will give 8” may be reformulated as an equation ‘*u* x 2 = 8’ using a letter *u* for the unknown number; and solved by recounting 8 in 2s. So, an equation is solved by moving a number to the opposite side with the opposite calculation sign. Here, solving equations is just another name for recounting in icon-units.

Outside action	Inside prediction
How many 2s will give 8?	u x 2 = 8 = (8/2) x 2 so
To answer, we recount 8 in 2s	u = 8/2

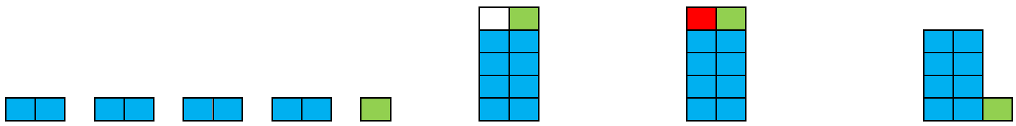
Figure 3. *An equation solved: to the opposite side with the opposite calculation sign*

Having pushed bundles away to stack, some unbundled may be left. So, from the total, we pull away the stack with a rope iconized as a horizontal stroke called subtraction.

Outside action	Inside prediction
From 9, 9/2 times 2s can be pushed away.	9/2 = 4.some
From 9, pull away 4 2s leaves 1	9 – 4 x 2 = 1
Prediction: T = 9 = 4B1 2s	Prediction: 9 = 4 x 2 + 1

Figure 4. *To predict unbundled, we pull away the stack from the original total*

Counting a total of 9 in 2s thus is predicted by the division 9/2 = 4.some. To predict leftovers, we pull away the stack of 4 2s from the total 9, predicted by saying ‘9-4x2’ giving the expected answer 1. So, a total of 9 may be counted in 2s as T = 9 = 4B1 2s, or as T = 4 x 2+1. Here a cross called addition iconizes the two ways to place the unbundled: next-to the stack iconized by a dot named a decimal point, T = 9 = 4.1 2s; or on-top of the stack counting in bundles as 1 = (1/2) x 2 giving T = 9 = 4B½ 2s, or counting what is missing in having a full bundle, T = 9 = 5B-1 2s, thus reifying fractions and negatives.



T = 9 = 4B 2s + 1 = 4B½ 2s = 5B-1 2s = 4B1 2s = 4.1 2s

Figure 5. *Unbundled singles may be placed on-top or next-to the stack of bundles*

Likewise, when counting in tens:

T = 48 = 4B tens + 8 = 4B8/10 tens = 5B-2 tens = 4B8 tens = 4.8 tens

Changing bundles to unbundled or vice versa gives ‘flexible bundle-numbers’ with or without an overload, or with an underload: T = 48 = 4B8 tens = 3B18 tens = 5B-2 tens.

2.7. Reflections on how to Recount into Tens

Once counted, a total may be recounted in the same unit, in a different unit, from tens into icons, or into tens. The first three cases are described in the chapter above.

Recounting from icons into tens, the recount formula cannot be used since there is no ten button or bundle button on the calculator. However, multiplication gives the result directly, only without units and with the decimal point moved one place.

Question: T = 3 8s = ? tens; answer: T = 3 8s = 3 x 8 = 24 = 2.4 tens

Recounting into tens includes multiplying two one-digit numbers, called setting up a multiplication table: a small table for the numbers 1-5, and a large for the numbers till 10.

2.7.1. Designing and Implementing a micro-curriculum

Turning over a stack will change e.g. 2 3s to 3 2s without changing the total. So, in multiplication, the order does not matter, the units may be commuted.

The small table follows directly from using fingers. It is obvious in the case of 2.

In the case of 3,

4x3 = 2x2x3 = 2 6s = 12 seeing a hand as a pawn with six extremities leaving it; and

5x3 = 3x5 = 3 5s = 3 hands = 1B5 = 15.

In the case of 4,

4x4 = 2x2x4 = 2 8s = 2 B-2s = 2B-4 = 1B6 = 16; and

5x4 = 4x5 = 4 hands = 2B0 = 20.

Finally, in the case of 5, 5x5 = 5 5s = 5 hands = 2B5 = 25.

In the large table we recount the numbers from 6 to 10 in bundles as B-4, B-3, etc.; and use a bead pegboard square with two rubber bands to show the actual stack as e.g.

6 7s = 6x7 = (B-4) x (B-3) = BB – 4B –3B + 4 3s removed twice = 3B12 = 4B2 = 42.

This roots the algebraic formula *(a – b)* x *(c – d)* = *a* x *c* – *a* x *d* – *b* x *c* + *b* x *d*.

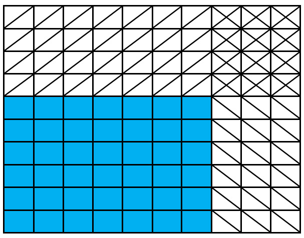


Figure 6. *A pegboard square shows that 6x7 = (B-4) x (B-3) = 10B – 4B – 3B + 4 x 3*

Recounting into tens also includes multiplying multi-digit numbers as e.g. 27 x 36 = 27 36s = ? tens. We may use a square or write the result in lines. It makes sense that changing the unit base from 36 to 10 will increase the height of the stack from 27 to 97.2.

Question: T = 27 x 36 = 27 36s = ? tens. Answer: T = 2B7 x 3B6 = (2B+7) x (3B+6) = 6BB+12B+21B+42 = 6BB + 33B + 4B2 = 6BB + 37B + 2 = 9BB7B2 = 972 = 97.2 tens.

Vice versa, recounting from tens includes division as the opposite of multiplication. Asking 16.8 tens is how many 7s thus gives the division 168/7. We may use the square bottom-up, or write the result in lines using flexible bundle-numbers. Here, it makes sense that changing the base from 10 to 7 will increase the height of the stack from 16.8 to 24.

Question: ? 7s = 16.8 tens.

Answer: *u* x 7 = 168; *u* = 168 /7 = 16B8 /7 = 14 B28 /7 = 2B 4 = 24. So 24 7s = 16.8 tens.

T = 27 x 36 = ? tens

	2B	7	
	6BB	21B	3B
	12B	42	6
6BB	33B	4B2	
6BB	37B	2	
9BB	7B	2	
9	7	2	

? x 7 = 168, or 168/7 = ?; answer: 2B4 = 24

		7	
		14B	? = 2B
		28	? = 4
	14B	28	
	16B	8	
1BB	6B	8	
1	6	8	

Figure 7. *A 2D schema used top-down for multiplication and bottom-up for division*

2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions

On a Lego brick we can double-count the dots and the rows, e.g. giving 2 rows per 8 dots on a 2x4 brick, thus producing the 'per-number' 2r/8d, or 2/8 r/d.

Double-counting a basket filled with 3red per 5 apples, the like units make the per-numbers a fraction, both being operators needing numbers to become numbers.

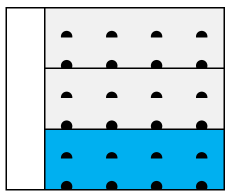
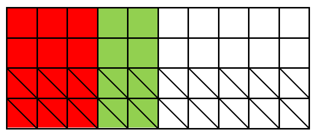
Asking "6 rows gives how many dots?", we recount 6 rows in the per-number as: T = 6r = (6/2) x 2r = (6/2) x 8d = 24d Likewise, when asking "how many rows gives 56 dots?": T = 56d = (56/8) x 8d = (56/8) x 2r = 14r	
To find the number of red apples among 20 apples, we set up a per-number equation *u*/20 = 3/5, or recount 20 in 5s: *u*/20 = 3/5; so *u* = 3/5 x 20 = 12. 20 a = (20/5) x 5a giving (20/5) x 3r = 12 red apples	

Figure 8. *Double-counting in two units creates per-numbers bridging the units*

Likewise with percent. 5% of 40 asks '5 per 100 is what per 40', giving the equation $u/40 = 5/100$; solved by moving 40 to opposite side with opposite sign: $u = 5/100 \times 40 = 2$.

The unit-changing recount formula may also be used on units as e.g.: $ = ($/kg) x kg. Thus with the per-number 2$/3kg, we may ask '?$ = 12kg' and '10$ = ?kg'. Recounting then gives: $ = ($/kg) x kg = 2/3 x 12 = 8; and kg = (kg/$)x$ = 3/2 x 10 = 15.

2.9. Reflections on how to De-model Trigonometry

In an $a \times b$ rectangle halved by its diagonal c, double-counting the sides creates the per-numbers sin $A = a/c$, cos $A = b/c$, and tan $A = a/b$. Filling a circle from the inside, we find a formula for the per-number perimeter per radius: $\pi = n \times \tan(180/n)$ for n large.

2.10. Reflections on how to Add Next-to and On-top, and how to Add Per-numbers

Once counted or recounted, blocks may be added next-to or on-top. Here adding 2 3s and 4 5s next-to as 8s means adding by areas, called integral calculus. Whereas adding 2 3s and 4 5s on-top means making the units like, using the recount formula to change units.

Adding 2kg at 3$/kg with 4kg at 5$/kg, the unit-numbers add directly as 2+4 = 6, whereas the per-numbers add by their areas (2x3 + 4x5)$ / 6kg, which is integral calculus where multiplication precedes addition. Vice versa, asking what to add to 4kg at 5$/kg to have 6 kg at 4$/kg, we subtract the initial block 4x5 from 6x4 before counting in 2s, which is differential calculus where subtraction precedes division.

3. Results and Discussion

This study asked: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

The answer is yes, if we de-model digits as icons with as many sticks as they represent if written less sloppy; if we use the flexible bundle-numbers children develop when adapting to Many; if we de-model operations as means for bundle-counting 8 as 8/2 2s, leading directly to the recount formula $T = (T/B) \times B$, used to change units, and to solve the question 'How many 2s in 8?' by recounting 8 in 2s: $u \times 2 = 8 = 8/2 \times 2$, so $u = 8/2$.

The operations are de-modeled as bundle-counting where division pushes away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to find unbundled singles to be placed next-to or on-top of the stack as decimal numbers, negative numbers or fractions; and later added with other stacks next-to as integral calculus, or on-top after making the units the same by using the proportionality of the recount formula.

Exemplifying assigns concrete meaning to abstract concepts, so de-modeling and reifying needs no psychological learning theories about how meaning is constructed.

As expected, concrete meaning makes mathematics easy to learn, as confirmed when tested in pilot projects in preschool, special education, and in adult and migrant education.

Of course, the effect of using flexible bundle-numbers and recounting operations should be studied in detail in other cases also to open up a completely new research paradigm (Kuhn, 1962), that may make obsolete all single-number material and research on mathematics in its grammar-based form before undergoing a communicative turn.

4. Conclusion

As to the questions asked in the introduction, the answers are: Yes, mathematics is hard if taught as pre setcentric mathe-matism true inside but seldom outside classrooms; and if taught as present setcentric meta-matism that by defining its concepts by abstract self-reference forces learners to construct a meaning themselves. And no, mathematics is not hard if taught as post setcentric many-math, allowing learners to further develop what they bring to school, a quantitative competence created by adaption to outside quantity.

When writing a mathematics curriculum, we must ask: Is its goal to master inside mathematics first as the only means to master outside Many later? Because then Vygotskian constructivism is needed to assign meaning to what is made meaningless by abstract self-reference or by tradition. Or is its goal to master outside Many directly, or via other means if the mathematics tradition is hindering 1 out of 4 or 5 students in acquiring basic numeracy? Because then Piagetian constructivism is better suited to allow students to accommodate existing schemas created by natural adaption to outside numerosity.

As to research, maybe the time has come to reread the Marx inscription in the entrance hall at the Humboldt mother-university in Berlin: “The philosophers have hitherto only interpreted the world in various ways. The point, however, is to change it.”

Changing from abstract single- to concrete double-numbers, mathematics education will meet The Universal Declaration of Human Rights Article 26, saying “Everyone has the right to education. (..) Education shall be directed to the full development of the human personality”; as well as Article 4, saying “No one shall be held in slavery or servitude; slavery (..) shall be prohibited in all their forms.” This also applies to the slavery of an abstract-referring mathematics wanting to colonize children’s own number-language.

Likewise, mathematics education should respect the UN Global Goals for Sustainable Development, where goal 4 about quality education writes in target 4.6 on universal literacy and numeracy: “By 2030, ensure that all youth and a substantial proportion of adults, both men, and women, achieve literacy and numeracy.”

So why teach children differently, if they already know?

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MÔ HÌNH HÓA NGƯỢC VỀ SỐ, PHÉP TOÁN VÀ PHƯƠNG TRÌNH: TỪ HIỂU BIẾT NỘI TẠI – NỘI TẠI ĐẾN HIỂU BIẾT NGOẠI VI – NỘI TẠI

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TÓM TẮT

Nhằm thích nghi với việc đếm Số nhiều trong cuộc sống, trẻ em sẽ kiến tạo tên các số (phù hợp với xã hội) bên trong trí não, nhưng họ làm điều đó như thế nào khi đếm số lượng? Để làm chủ được việc đếm Số nhiều, trẻ em sử dụng những số lượng bó với đơn vị xác định (ví dụ: bó 2 vật, bó 10 vật...); và mềm dẻo sử dụng các phân số, số thập phân và số âm cho đơn vị. Bài báo đề xuất một cách thiết kế chương trình dạy học bằng cách thay thế tâm lí dựa trên sự trừu tượng bởi tâm lí dựa trên sự cụ thể làm trung gian cho việc hiểu số thông qua sự mô hình hóa ngược từ các kiến thức toán học cốt lõi, vì thế cho phép học sinh mở rộng ngôn ngữ về số trước khi họ đến trường.

Từ khóa: số; phép tính; phương trình; năng lực về số; tỉ lệ; tuổi thơ