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EXISTENCE RESULTS FOR A CLASS OF LOGISTIC SYSTEMS

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ABSTRACT

We consider logistic system

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & \text{in } \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that the nonlinearity f_i, g_i *satisfies certain growth condition. Using the fixed point index and the arguments on monotone minorant, we prove the existence results for the system. This extends some known results.*

Keywords: Logistic system, *(p-1)* **-** sublinear, fixed point index.

TÓM TẮT

Sự tồn tại nghiệm của một lớp hệ phương trình logistic

Trong bài báo này, chúng tôi xét hệ phương trình logistic sau:

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & \text{trong } \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & \text{trong } \Omega, \\ u = v = 0 & \text{trên } \partial\Omega, \end{cases}$$

Giả sử các hàm phi tuyến f_i, g_i *thỏa mãn điều kiện về bậc tăng (của ẩn hàm) được chỉ ra sau. Bằng phương pháp bậc tô pô kết hợp với lí luận về chặn dưới đơn điệu, chúng tôi chứng minh sự tồn tại nghiệm cho hệ. Đây là một kết quả mở rộng cho các nghiên cứu trước đây.*

Từ khóa: hệ phương trình logistic, (p-1)-tuyến tính, bậc tô pô.

1. Introduction

In this paper, we consider the following system

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & \text{in } \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

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where u,v are non-trivial non-negative unknown functions, $\Omega\subset\mathbb{R}^N (N\geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1<p<N$, $\lambda>0$ is a real parameter and $f_i, g_i, i=1,2$ are the suitable functions.

In the special case, when

$$f_1(x,u,v)=u+b(x)uv,\ f_2(x,u,v)=u+c(x)uv,\ g_1(x,u)=u^2,\ g_2(x,v)=v^2$$

the Problem (1.1) is the symbolic Lotka - Volterra model with diffusion and transport effects and was studied in [3]. We refer the reader to the papers [4, 7] and to [3, 8] and references therein for more imformations on the logistic equations and logistic systems, respectively.

In [8], G. Y. Yang and M. X. Wang have extended the study in [3] and considered the system

$$\begin{cases} -\Delta_p u=\lambda|u|^{p-2}u+b(x)|u|^{p-2}uv-f(u) & \text{in } \Omega,\\ -\Delta_q v=\mu|v|^{q-2}v+c(x)|v|^{q-2}vu-g(v) & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $b(x),c(x)$ are positive continuous functions and $f,g\in C^1[0,\infty]$ satisfy the following restricted conditions

i. the functions $F(s)=\dfrac{f(s)}{s^{p-1}}, G(s)=\dfrac{g(s)}{s^{p-1}}$ are positive stricly increasing,

ii. there exist positive numbers k_1,k_2,M such that

$k_1 s\leq F(s)\leq k_1 s+M, k_2 s\leq G(s)\leq k_2 s+M.$

Thus, the functions in the right - hand side of our Problem (1.1) are more general than the functions in (1.2). On the other hand, our method of studying can be applies to the case, when the operator Δ_p and the parameter λ in second equation of the System (1.1) are replaced with Δ_q and μ, respectively.

In this paper, we consider only the case of $(p-1)$- sublinear growth for the second variable in the functions $f_i, i=1,2$. The cases of $(p-1)$- linear and $(p-1)$- superlinear growth will be considered in a future paper.

2. Preliminary results

2.1. Equations in ordered spaces

Let E be a Banach space ordered by the cone $K\subset E$, that is, K is a closed convex subset such that $\lambda K\subset K$ for all $\lambda\geq 0$, $K\cap(-K)=\{\theta\}$ and ordering in E is defined by $x\leq y$ *iff* $y-x\in K$.

If D is a bounded relatively open subset of K and $F:\bar{D}\to K$ is a compact operator such that $F(u)\neq u, \forall u\in\partial D$, then the fixed point index $i(F,D,K)$ of F on D with respect to K is well-defined. This fixed point index admits all usual properties of the Leray - Schauder degree (see e.g [6]). In particular, we have the following important results on computation of the index.

Proposition 2.1. *Assume that D is a bounded relatively open subset of K and $F:\bar{D}\to K$ is a compact operator satisfying $F(u)\neq u, \forall u\in\partial D$. If there exits $u_0\in K$, $\{\theta\}$ such that*

$$u\neq F(u)+tu_0, \forall t>0, \forall u\in\partial D,$$

then $i(F,D,K)=0$.

Proposition 2.2. *Let (E,K) and (E_1,K_1) be the ordered Banach spaces and $N:K\to K_1$ be a continuous, bounded operator, $P:K_1\to K$ be a compact operator, $P(\theta)=\theta$. Let $D\subset E$ be a bounded open subset containing θ. If*

$$u\neq P[tN(u)], \forall t\in[0,1], \forall u\in\partial D\cap K,$$

then $i(PoN,D,K)=1$.

Here, we use the notation $i(PoN,D,K)$ instead of $i(PoN,D\cap K,K)$.

2.2. A reduction to the fixed point equations

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary, $1<p<N$. We denote the norms in the spaces $W_0^{1,p}(\Omega)$ and $L^t(\Omega)$ by $\|.\|$ and $\|.\|_t$ respectively. In these spaces, we consider the order cone of nonnegative functions. In order to reduce the boundary value Problem (1.1) to a fixed point equation in an ordered Banach space, we need the following result [7].

Theorem 2.3. *Assume that the Caratheodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions*

(g1) $g(x,0)=0$, and $g(x,u)$ is an increasing function with respect to the variable u for a.e $x\in\Omega$,

(g2) there exist $a\in\mathbb{R}^+, 0<\beta<p^*-1$ and $b\in L^{(\beta+1)'}(\Omega)$ such that $|g(x,u)|\leq a|u|^{\beta}+b(x)$ for $(x,u)\in\Omega\times\mathbb{R}$.

Then, for any $h\in W^{-1,p'}(\Omega)$, there exists a unique function $u\in W_0^{1,p}(\Omega)$ such that $g(x,u)\in L^{(p^*)'}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi+\int_\Omega g(x,u)\varphi=\langle h,\varphi\rangle, \ \forall\varphi\in W_0^{1,p}(\Omega) \quad (1.3)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Definition 2.4. Let $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function, that is, $f_i(\cdot, u, v)$ is measurable for all $(u,v) \in \mathbb{R} \times \mathbb{R}$ and $f_i(x, \cdot, \cdot)$ is continuous for a.e $x \in \Omega$. We say that the pair $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ is a weak solution of the system

$$\begin{cases} -\Delta_p u = f_1(x,u,v) & \text{in } \Omega, \\ -\Delta_p v = f_2(x,v,u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

if $f_1(x,u,v),\ f_2(x,v,u) \in L^{(p^*)'}(\Omega)$ and

$$\begin{cases} \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi = \int_\Omega f_1(x,u,v)\varphi \\ \int_\Omega |\nabla v|^{p-2} \nabla v . \nabla \phi = \int_\Omega f_2(x,v,u)\phi, \end{cases} \quad \forall \varphi, \phi \in W_0^{1,p}(\Omega).$$

Let the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be defined by

$$\langle Au, \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi, \forall u, \varphi \in W_0^{1,p}(\Omega).$$

Then we have the following results (see [2, 5]).

Proposition 2.5.

1. The mapping $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous and of type S^+, that is, for every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$u_n \to u$ *weakly, and* $\limsup_n \langle Au_n, u_n - u \rangle \le 0,$

we have $u_n \to u$ strongly.

2. If $u, v \in W_0^{1,p}(\Omega)$ satisfying $\langle Au - Av, (u-v)^+ \rangle \le 0$, then $u \le v$ a.e. in Ω. Here, $u^+ = \max\{u, \theta\}$.

3. The inverse operator A^{-1} is compact from $L^\infty(\Omega)$ into $C_0^1(\bar{\Omega})$.

Proposition 2.6. (see [7]) *The operator $P : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ that assigns each $h \in W^{-1,p'}(\Omega)$ the unique solution of problem (2.3) has the following properties:*

1. P is increasing in the sense that $h_1 \le h_2$ implies $P(h_1) \le P(h_2)$. Here, $h_1 \le h_2$ means that $\langle h_2 - h_1, u \rangle \ge 0, \forall u \in W_0^{1,p}(\Omega), u \ge 0$.

2. P is continuous and $P(M)$ is bounded if M is bounded (we also say that P is a bounded operator).

3. *If $\delta > (p^*)'$, then P is compact from $L^\delta(\Omega)$ into $W_0^{1,p}(\Omega)$.*

Proposition 2.7. *Let* $f:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *be a Caratheodory function and let* N_f *be the associated Nemytskii operator defined by* $N_f(u,v)(x)=f(x,u(x),v(x))$ *for all* $u,v\in W_0^{1,p}(\Omega)$.

1. *Assume that*

$$|f(x,u,v)|\le g(x,|u|,|v|),$$

where $g:\Omega\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ *is a Caratheodory function which is nondecreasing with respect to the second and the third variables, and satisfies the following condition*

$$u,v\in L^{p^*}(\Omega),u,v\ge\theta\Rightarrow g(\cdot,u,v)\in L^{\delta}(\Omega) \tag{1.4}$$

for some $\delta\in(1,\infty)$. *Then, the Nemytskii operator* N_f *is continuous from* $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ *into* $L^{\delta}(\Omega)$.

2. *Assume that the function* f *satisfies*

$$|f(x,u,v)|\le m(x)|u|^{\alpha}+n(x)|v|^{\gamma}, \quad (f)$$

where $\alpha,\gamma<p^*-1$ *and* $m\in L^q(\Omega),n\in L^r(\Omega)$ *with* $q>(\frac{p^*}{1+\alpha})',r>(\frac{p^*}{1+\gamma})'$

Then the Nemytskii operator N_f *is continuous and bounded from* $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ *into* $L^{\delta}(\Omega)$ *with* $\delta=\min\left\{\frac{qp^*}{q\alpha+p^*},\frac{rp^*}{r\gamma+p^*}\right\}>(p^*)'$.

Proof 1. Assume that $u_n\to u_0,v_n\to v_0$ in $W_0^{1,p}(\Omega)$, we shall prove that some subsequence of $N_f(u_n,v_n)$ converges to $N_f(u_0,v_0)$.

Passing to a subsequence if necessary, we may assume $u_n\to u_0,v_n\to v_0$ a.e in Ω and and there exist $u,v\in L^{p^*}(\Omega)$ such that

$$|u_n(x)|\le u(x),|v_n(x)|\le v(x) \quad \text{a.e. in } \Omega.$$

Then we have $N_f(u_n,v_n)\to N_f(u_0,v_0)$ a.e. in Ω and $|N_f(u_n,v_n)|\le g(x,u(x),v(x))\in L^{\delta}(\Omega)$. This along with the Dominated Convergence Theorem yields $N_f(u_n,v_n)\to N_f(u_0,v_0)$ in $L^{\delta}(\Omega)$.

2. For $u,v\in L^{p^*}(\Omega),u,v\ge 0$, we have $m(x)u^{\alpha}\in L^{\frac{qp^*}{q\alpha+p^*}}(\Omega),n(x)v^{\gamma}\in L^{\frac{rp^*}{r\gamma+p^*}}(\Omega)$ which implies $m(x)u^{\alpha}+n(x)v^{\gamma}\in L^{\delta}(\Omega)$. Therefore, N_f is continuous by the first assertion. The boundedness of the operator N_f follows from

$$\|N_f(u,v)\|_{\delta}\le c\left(\|m\|_q\|u\|_{\alpha q'}^{\alpha}+\|n\|_r\|v\|_{\gamma r'}^{\gamma}\right)\le c\left(\|m\|_q\|u\|^{\alpha}+\|n\|_r\|v\|^{\gamma}\right).$$

Corollary 2.8. *If the Caratheodory functions* $g:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$, $f:\Omega\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ *satisfy conditions* $(g1)$, $(g2)$, (f), *then the operator* PoN_f *is compact from* $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ *into* $W_0^{1,p}(\Omega)$.

Now, we reduce the Problem 1.1 to the fixed point problem. Denote by N_{f_i} the Nemytskii operators associated to $f_i, i=1,2$ and by P_i the operators defined in Proposition 2.3 for $g_i, i=1,2$. It is clearly that if $f_i, g_i, i=1,2$ satisfy conditions $(g_1),(g_2),(f)$ then the mappings $P_i oN_{f_i}, i=1,2$ are compact from $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ into $W_0^{1,p}(\Omega)$. Let $N(u,v)=(N_{f_1}(u,v),N_{f_2}(u,v))$ and $PoN:=(P_1oN_{f_1},P_2oN_{f_2})$ then PoN is also compact from $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ into itself and $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ is a solution of Problem (1.1) if and only if $(u,v)=Po\lambda N(u,v)$.

3. The main results

Throughout this section, we always use C to denote a positive constant that is independent of the main parameters involved but whose values may differ from line to line. We consider the cone $K=\{(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega): u,v\ge\theta\}$ and by $\|(u,v)\|=\|u\|+\|v\|$, $\|(u,v)\|_p=\|u\|_p+\|v\|_p$ we denote the norms in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ and $L^p(\Omega)\times L^p(\Omega)$ respectively. Noting that, for any $t>0$, one has

$$\frac{1}{2}\left(\|u\|^t+\|v\|^t\right)\le\left(\|u\|+\|v\|\right)^t\le 2^t\left(\|u\|^t+\|v\|^t\right).(1.5)$$

Theorem 3.1. *Assume that the Caratheodory function* $g_i:\Omega\times\mathbb{R}^+\to\mathbb{R}^+, i=1,2$ *satisfies conditions* $(g1)$, $(g2)$ *in Section 2, and the Caratheodory function* $f_i:\Omega\times\mathbb{R}^+\times\mathbb{R}^N\to\mathbb{R}$, $i=1,2$ *satisfies:*

(H1) (a) $0\le f_i(x,z,t)\le m_i(x)z^\alpha+n_i(x)t^\gamma$, *where* $\alpha<p-1, m_i(x)\in L^q(\Omega)$, *with* $q>\left(\frac{p^*}{1+\alpha}\right)'$, $i=1,2$

(b) $f_i(x,z,0)=0, \forall(x,z)\in\Omega\times\mathbb{R}^+, i=1,2.$

(H2) At least one of the following conditions holds

(a) $\gamma<p-1, n_i(x)\in L^r(\Omega), r>\max\left\{(p-\gamma)',\left(\frac{pp^*}{\gamma p+p^*}\right)\right\}, i=1,2$;

(b) $\gamma < \dfrac{(\beta+1)(p-1)}{p}$, $g_i(x,t) \ge a_i t^{\beta} - b_i(x)$, *where* β *and* $b_i(x)$ *are as in condition (g2)*, $i = 1,2$ *and* $n_i(x) \in L^r(\Omega), r > \left(\dfrac{p(\beta+1)}{\gamma p + \beta + 1}\right)'$.

(H3) There exist an open subset $\Omega_0 \Subset \Omega$, *and positive numbers* $\delta < \alpha, \varepsilon, m_0, n_0, l_0$ *such that*

$$f_i(x,z,t) \ge m_0 z^{\alpha-\delta} t^{\delta},\ g_i(x,z) \le n_0 u^{\alpha+\varepsilon},\ \forall (x,z,t) \in \Omega_0 \times [0,l_0] \times \mathbb{R},\ i = 1,2.$$

Then, for all $\lambda > 0$ *the Equation (1.1) has a non-negative solution* (u,v) *satisfying* $u,v \ge \theta$ *and* $u \not\equiv \theta, v \not\equiv \theta$.

Proof. For the sake of simplicity, we shall put $\lambda = 1$ and write $N(u,v)$ instead of $\lambda N(u,v)$. We split the proof into several steps.

Step 1. We shall prove that there is a sufficiently large number R such that

$$(u,v) \ne P[tN(u,v)],\ \forall t \in [0,1],\ \forall u,v \ge \theta,\ \|(u,v)\| = R.$$

Assume in the contrary that there exist sequences $\{t_n\} \subset [0,1]$, and $u_n, v_n \ge \theta$, $\|(u_n,v_n)\| \to \infty$ such that $(u_n,v_n) = P[t_n N(u_n,v_n)]$, or equivalently, one has

$$\begin{cases} \langle Au_n, \varphi\rangle + \int_\Omega g_1(x,u_n)\varphi = \int_\Omega t_n f_1(x,u_n,v_n)\varphi \\ \langle Av_n, \phi\rangle + \int_\Omega g_2(x,v_n)\phi = \int_\Omega t_n f_2(x,v_n,u_n)\phi \end{cases} \quad \forall \varphi,\phi \in W_0^{1,p}(\Omega). \tag{1.6}$$

Choosing $\varphi = u_n, \phi = v_n$ in (1.6) and using $(H1)$ we obtain

$$\begin{cases} \|u_n\|^p + \int_\Omega g_1(x,u_n)u_n \le \int_\Omega m_1(x)u_n^{1+\alpha} + \int_\Omega n_1(x)v_n^{\gamma}u_n, \\ \|v_n\|^p + \int_\Omega g_2(x,v_n)v_n \le \int_\Omega m_2(x)v_n^{1+\alpha} + \int_\Omega n_2(x)u_n^{\gamma}v_n. \end{cases} \tag{1.7}$$

In the case that condition (a) in $(H1)$ holds, by adding sides by sides of the inequalities in (1.7), we have

$$C.\ \|(u_n,v_n)\|^p \le \int_\Omega m_1(x)u_n^{1+\alpha} + \int_\Omega m_2(x)v_n^{1+\alpha} + \int_\Omega n_1(x)v_n^{\gamma}u_n + \int_\Omega n_2(x)u_n^{\gamma}v_n. \tag{1.8}$$

By Holder's inequality, Young's inequality and some simple computations we obtain

$$\begin{aligned} \|(u_n,v_n)\|^p \le\ & C\left(\|m_1\|_q + \|m_2\|_q\right)\|(u_n,v_n)\|_{(1+\alpha)q'}^{1+\alpha} + \\ & C\left(\|n_1\|_r + \|n_2\|_r\right)\left[\varepsilon\|(u_n,v_n)\|_p^{p/r'} + C(\varepsilon)\|(u_n,v_n)\|_s^{s/r'}\right], \end{aligned} \tag{1.9}$$

here $s = \gamma\left(\dfrac{p}{r'}\right)'.r'$ and we have used the inequality $(a+b)^{\theta} < a^{\theta} + b^{\theta},\ \forall a,b > 0,\ \forall \theta \in (0,1)$.

It follows from $(1+\alpha)q', s < p^*$ and (1.9) that

$$\|(u_n,v_n)\|^p \leq C\Big(\|(u_n,v_n)\|^{1+\alpha} + \|(u_n,v_n)\|^{s/r'}\Big),$$

which contradicts to $\|(u_n,v_n)\| \to \infty$ and $1+\alpha < p, \frac{s}{r'} < p$.

Next we consider the case (b) in (H1). Adding sides by sides of the inequalities in (1.7), we deduce

$$\|(u_n,v_n)\|^p + C\|(u_n,v_n)\|_{\beta+1}^{\beta+1} \leq C\left(\|m_1\|_q + \|m_2\|_q\right)\|(u_n,v_n)\|_{(1+\alpha)q'}^{1+\alpha} + \int_\Omega \left[n_1(x)v_n^\gamma u_n + n_2(x)u_n^\gamma v_n\right] + \int_\Omega \left[b_1(x)u_n + b_2(x)v_n\right] \tag{1.10}$$

By the Holder's and Young's inequalities we have

$$\int_\Omega \left[b_1(x)u_n + b_2(x)v_n\right] \leq C\varepsilon\|(u_n,v_n)\|_{\beta+1}^{\beta+1} + C(\varepsilon)\int_\Omega \left[(b_1(x))^{(\beta+1)'} + (b_2(x))^{(\beta+1)'}\right], \tag{1.11}$$

$$\int_\Omega \left[n_1(x)u_n^\gamma v_n + n_2(x)v_n^\gamma u_n\right] \leq C\left[\|n_1\|_r + \|n_2\|_r\right]\left[\varepsilon\|(u_n,v_n)\|_p^{p/r'} + C(\varepsilon)\|(u_n,v_n)\|_s^{s/r'}\right] \tag{1.12}$$

where $s = \gamma\left(\frac{p}{r'}\right)' r'$.

From (1.10), (1.11), (1.12) it follows that

$$\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \leq C\Big(\|(u_n,v_n)\|_{(1+\alpha)q'}^{1+\alpha} + \|(u_n,v_n)\|_s^{s/r'} + 1\Big). \tag{1.13}$$

Since $(1+\alpha)q' < p^*, 1+\alpha < p$, (1.13) implies

$$\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \leq C\|(u_n,v_n)\|_s^{s/r'}. \tag{1.14}$$

Since $s < \beta+1$ we deduce from (1.14) that

$$\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \leq C\|(u_n,v_n)\|_{\beta+1}^{s/r'}$$

which yields that $\|(u_n,v_n)\|_{\beta+1} \to \infty$ and that is a contradiction because $\frac{s}{r'} < \beta+1$.

Step 2. We claim that there exists a sufficiently small number $r > 0$ such that

$$(u,v) \neq P[N(u,v)] + t(u_0,u_0), \forall t > 0, \forall u,v \geq \theta, \|(u,v)\| = r, \tag{1.15}$$

where the functions u_0 is given as follows. Let $\bar{u}$ be the positive eigenfunction corresponding to the principal eigenvalue λ_0 of the problem

$$-\Delta_p u(x) = \lambda |u|^{p-2} u \text{ in } \Omega_0, \ u(x) = 0, \text{ on } \partial\Omega_0.$$

Then we put $u_0 = c\bar{u}$ in $\Omega_0, u_0 = 0$ in Ω, Ω_0 where c is sufficiently small number.

It is proved in [1] that

$$\langle Au_0, \varphi\rangle \leq \int_\Omega u_0^\alpha \varphi, \forall \varphi \in W_0^{1,p}(\Omega), \varphi \geq 0 \tag{1.16}$$

Before proving (1.15) we need some preliminary results. We define the function k by setting

$$k(x,u,v)=\begin{cases} m_0 u^{\alpha-\delta}v^{\delta} & \text{if } x\in\Omega_0, \\ 0 & \text{if } x\in\Omega\,,\ \ \Omega_0 \end{cases} \tag{1.17}$$

Denote by N_1 the Nemytskii operator corresponding to k. We will show that for a sufficiently small $t>0$ and for a number σ such that $1>\sigma>\max\left\{\dfrac{\alpha}{p-1};\dfrac{\alpha}{\alpha+\varepsilon}\right\}$ one has

$$P_1 o N_{f_1}(tu_0,tu_0)\geq (P_1 o N_1(tu_0,tu_0)\geq t^{\sigma}u_0 \tag{1.18}$$

and

$$P_2 o N_{f_2}(tu_0,tu_0))\geq P_2 o N_1(tu_0,tu_0))\geq t^{\sigma}u_0. \tag{1.19}$$

We will only prove (1.18), assertion (1.19) is proved similarly. Putting $w=P_1 o N_1(tu_0,tu_0)$, we have by definition of P_1 that

$$\langle Aw,\varphi\rangle=\int_{\Omega}\big[k(x,tu_0,tu_0)-g_1(x,w)\big]\varphi,\ \forall\varphi\in W_0^{1,p}. \tag{1.20}$$

Taking $\varphi=(t^{\sigma}u_0-w)^{+}$ in (1.16), (1.20) we easily deduce that

$$\begin{aligned}\langle A(t^{\sigma}u_0)-Aw,(t^{\sigma}u_0-w)^{+}\rangle &\leq \int_{\Omega_1}\left[\lambda_0 t^{\sigma(p-1)}u_0^{\alpha}+g_1(x,w)-k(x,tu_0,tv_0)\right](t^{\sigma}u_0-w)\\ &=\int_{\Omega_1}\left[\lambda_0 t^{\sigma(p-1)}u_0^{\alpha}+g_1(x,w)-m_0(tu_0)^{\alpha}\right](t^{\sigma}u_0-w):=\int_{\Omega_1}h,\end{aligned} \tag{1.21}$$

where $\Omega_1=\{t^{\sigma}u_0\geq w\}$.

It is easy to see that $h\leq 0$ in Ω_0 , Ω_1. On the other hand, in $\Omega_0\cap\Omega_1$ we have

$$\begin{aligned}h&\leq\left[\lambda_0 t^{\sigma(p-1)}u_0^{\alpha}-m_0(tu_0)^{\alpha}+n_0(t^{\sigma}u_0)^{\alpha+\varepsilon}\right](t^{\sigma}u_0-v)\\ &=(tu_0)^{\alpha}\left[\lambda_0 t^{\sigma(p-1)-\alpha}-m_0+n_0 t^{\sigma(\alpha+\varepsilon)-\alpha}u_0^{\varepsilon}\right](t^{\sigma}u_0-v)\end{aligned}$$

Therefore, by the bounded-ness of u_0, we have $h\leq 0$ in Ω_1 provided that t is sufficiently small.

Consequently, $\langle A(t^{\sigma}u_0)-Aw,(t^{\sigma}u_0-w)^{+}\rangle\leq 0$ which implies $t^{\sigma}u_0\leq w$. The first inequality in (1.18) holds by the increasingly of the operator P_1. Hence, (1.18) is proved.

We now prove that (1.15) holds. Assume by contradiction that we can find $t_n>0$, $u_n,\ v_n\geq\theta,\ \ \|(u_n,v_n)\|\to 0$ such that

$$(u_n,v_n)=PoN(u_n,v_n)+t_n(u_0,u_0). \tag{1.22}$$

Then we have $(u_n,v_n)\geq t_n(u_0,u_0)$, and we denote by s_n the maximal number such that $(u_n,v_n)\geq s_n(u_0,u_0)$. We have $s_n>0$ and $s_n\to 0$ (note $s_n\geq t_n$, and $C\ \|(u_n,v_n)\|\geq \|(u_n,v_n)\|_{p^*}\ \geq\ s_n\ \|(u_0,u_0)\|_{p^*}$).

From (1.18), (1.19), (1.22) it follows that

$$(u_n,v_n)\geq P[N(u_n,v_n)]\geq (P_1oN_1(u_n,v_n),P_2oN_1(u_n,v_n))$$
$$\geq (P_1oN_1(s_n(u_0,u_0)),P_2oN_1(s_n(u_0,u_0))\geq s_n^{\sigma}(u_0,u_0).$$

This, by definition of s_n, yields $s_n^{\sigma}\leq s_n$ which is a contradiction to that $\sigma<1, s_n\to 0$

Step 3. From Steps 1, 2 and Propositions 2.1, 2.2 we get

$$i(PoN,B((\theta,\theta),R),K)=1, \text{ for large } R,$$

and

$$i(PoN,B((\theta,\theta),r),K)=0, \text{ as } r \text{ is small.}$$

Therefore, there exists $(u,v)\geq(\theta,\theta)$ such that $r\leq\ \|(u,v)\|\ \leq R$ and $(u,v)=PoN(u,v)$. This means that the Problem (1.1) has a positive solution.

Finally, we prove that this solution (u,v) satisfies $u\not\equiv\theta$ and $v\not\equiv\theta$. Indeed, if $u\equiv\theta$ then by assumption $f_2(x,v,0)=0$ we have

$$-\Delta_p v+g_2(x,v)=\theta$$

which implies that $v\equiv\theta$, a contradiction.

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