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Research Article

STRONG CONVERGENCE OF A HYBRID ITERATION FOR A GENERALIZED MIXED EQUILIBRIUM PROBLEM AND A BREGMAN TOTALLY QUASI-ASYMPTOTICALLY NONEXPANSIVE MAPPING IN BANACH SPACES

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ABSTRACT

The purpose of this paper is to combine the Bregman distance with the shrinking projection method to introduce a new hybrid iteration process for a generalized mixed equilibrium problem and a Bregman totally quasi-asymptotically nonexpansive mapping. After that, under some suitable conditions, we prove that the proposed iteration strongly converges to the Bregman projection of the initial point onto the common element set of the solution set of a generalized mixed equilibrium problem and the fixed point set of a Bregman totally quasi-asymptotically nonexpansive mapping in reflexive Banach spaces. This theorem extends and improves the results reported by Alizadeh and Moradlou (2016) from a generalized hybrid mapping and an equilibrium problem in Hilbert spaces to a Bregman totally quasi-asymptotically nonexpansive mapping and a generalized mixed equilibrium problem in reflexive Banach spaces. The result is applied to a generalized mixed equilibrium problem and a Bregman quasi-asymptotically nonexpansive mapping in reflexive Banach spaces. In addition, an example is provided to illustrate the proposed iteration process.

Keywords: Bregman totally quasi-asymptotically nonexpansive mapping; generalized mixed equilibrium problem; hybrid iteration process; reflexive Banach spaces

1. Introduction

Suppose that X is a real reflexive Banach space, Ω is a nonempty, closed, and convex subset of $X,$ and X^* is a dual space of $X.$ Let $f:\Omega\times\Omega\to\mathbb{R},$ $\varphi:\Omega\to\mathbb{R}$ be two functions and $\psi:\Omega\to X^*$ be a mapping. We denote the value of $u^*\in X^*$ at $u\in X$ by $\langle u^*,u\rangle.$ The *generalized mixed equilibrium problem* (GMEP) is to find $u\in\Omega$ such that

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$f(u,v)+\langle\psi(u),v-u\rangle+\varphi(v)\geq\varphi(u)$ for all $v\in\Omega$. The set of solutions of (GMEP) is denoted by $GMEP(f,\varphi,\psi)=\{u\in\Omega: f(u,v)+\langle\psi(u),v-u\rangle+\varphi(v)\geq\varphi(u),\forall v\in\Omega\}$. Note that, if $\varphi\equiv 0$ and $\psi\equiv 0$, the problem (GMEP) is reduced into the equilibrium problem (EP) which is to find $u\in\Omega$ such that $f(u,v)\geq 0$ for all $v\in\Omega$.

In recent times, there were many methods for solving the above problems. In 2016, Darvish introduced an iterative method for finding common elements of the solutions set of the problem (GMEP) and the fixed points set of a Bregman strongly nonexpansive mapping in reflexive Banach spaces. In 2016, Zhu and Huang introduced a new hybrid iterative scheme for finding common solutions to the problem (EP) and fixed points of Bregman totally quasi-asymptotically nonexpansive mappings. In 2018, Ni and Wen proposed a new iterative scheme for finding a common solution of a system of the problem (GMEP) and fixed points of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings. Note that these convergence results extend and improve the existing results from Hilbert spaces or smooth Banach spaces to reflexive Banach spaces. Therefore, an interesting work naturally raised is to continue to generalize the existing convergence results from Hilbert spaces to reflexive Banach spaces.

In this paper, motivated by the iteration process proposed by Alizadeh and Moradlou (2016), we introduce a new hybrid iterative scheme which is to find common elements of the set of solutions of the problem (GMEP) and the set of fixed points of Bregman totally quasi-asymptotically nonexpansive mappings. After that, we prove a strong convergence theorem for the proposed iteration in reflexive Banach spaces. In addition, a numerical example is given to illustrate the results.

Now, we recall some notions and results which will be useful in what follows.

Assume that $g: X\to(-\infty,+\infty]$ is a lower semi-continuous, convex, and proper function. We denote the domain of g by $\mathrm{dom}g=\{u\in X: g(u)<+\infty\}$. For any $u\in\mathrm{int}(\mathrm{dom}g)$ and $v\in X$, we denote by $g'(u,v)=\lim_{\lambda\to 0^+}\dfrac{g(u+\lambda v)-g(u)}{\lambda}$ (1.1) the right-hand derivative of g at u in the direction v. The function g is called *Gâteaux differentiable at* u if the limit (1.1) exists for all v. Then the gradient of g at u is $\nabla g(u)$, which is defined by $\langle\nabla g(u),v\rangle=g'(u,v)$ for all $v\in X$. The function g is called *Fréchet differentiable at* u if the limit (1.1) is attained uniformly in $\|v\|=1$. The function g is called be *uniformly Fréchet differentiable on a subset* Ω of X if the limit (1.1) is attained uniformly for $u\in\Omega$ and $\|v\|=1$.

Note that if g is uniformly Fréchet differentiable, then g is uniformly continuous (see Ambrosetti & Prodi, 1993, Theorem 1.8). If g is Gâteaux differentiable and lower semi-continuous convex, then g is bounded on bounded sets if and only if ∇g is bounded on bounded sets (see Ambrosetti & Prodi, 1993, Proposition 1.1.11). Furthermore, if g is uniformly Fréchet differentiable and bounded on bounded subsets, then ∇g is uniformly continuous on bounded subsets of X^* (see Reich & Sabach, 2009, Proposition 1).

Let $u \in \mathrm{int}(\mathrm{dom} g)$, the Fenchel conjugate of g is the function $g^* : X^* \to (-\infty, +\infty]$ defined by $g^*(u^*) = \sup\{\langle u^*, u\rangle - g(u) : u \in X\}$ for all $u^* \in X^*$.

Definition 1.1. (Chang *et al.*, 2014, Definition 2.2)**.** Let X be a real reflexive Banach space and $g : X \to (-\infty, +\infty]$ be a function. Then g is called *Legendre* if

(L1) $\mathrm{int}(\mathrm{dom} g) \neq \varnothing$, g is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} g)$ and $\mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom} g)$.

(L2) $\mathrm{int}(\mathrm{dom} g^*) \neq \varnothing$, g^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} g^*)$ and $\mathrm{dom}(\nabla g^*) = \mathrm{int}(\mathrm{dom} g^*)$.

Remark 1.2. (Chang *et al.*, 2014, Remark 2.3)**.** *Let X be a real reflexive Banach space and $g : E \to (-\infty, +\infty]$ be Legendre. Then*

(1) g is Legendre if and only if g^* is Legendre.

(2) $\nabla g = (\nabla g^*)^{-1}$, $\mathrm{ran}(\nabla g) = \mathrm{dom}(\nabla g^*)$ and $\mathrm{ran}(\nabla g^*) = \mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom} g)$, where $\mathrm{ran}(\nabla g)$ is the range of ∇g.

Definition 1.3. (Censor & Lent, 1981, p.324)**.** Let X be a real reflexive Banach space and $g : X \to (-\infty, +\infty]$ be Gâteaux differentiable. Then $D_g : \mathrm{dom} g \times \mathrm{int}(\mathrm{dom} g) \to [0, \infty)$, defined by $D_g(u, v) = g(u) - g(v) - \langle \nabla g(v), u - v\rangle$ is called the *Bregman distance* with respect to g.

From the definition, we have $D_g(u, v) + D_g(v, w) - D_g(u, w) = \langle \nabla g(w) - \nabla g(v), u - v\rangle$ for all $u \in \mathrm{dom} g$ and $v, w \in \mathrm{int}(\mathrm{dom} g)$.

Let $g : X \to (-\infty, +\infty]$ be Gâteaux differentiable and $V_g : X \times X^* \to [0, \infty)$ be defined by $V_g(u, u^*) = g(u) - \langle u^*, u\rangle + g^*(u^*)$ for all $u \in X$ and $u^* \in X^*$.

Remark 1.4. *Let $g : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function. Then*

(1)(Kohsaka & Takahashi, 2005, Lemma 3.2) *For any $u \in X$ and $u^* \in X^*$, we have $V_g(u, u^*) = D_g(u, \nabla g^*(u^*))$.*

(2) (Kumam *et al.*, 2016, p.7) *V_f is convex in the second variable. Furthermore, for any $u \in \mathrm{dom} g$, $\{u_k\}_{k=1}^m \subset \mathrm{int}(\mathrm{dom} g)$ and $\{t_k\}_{k=1}^m \subset [0,1]$ with $\sum_{k=1}^m t_k = 1$, we have*

$$D_g(u, \nabla g^*(\sum_{k=1}^m t_k \nabla g(u_k))) \leq \sum_{k=1}^m t_k D_g(u, u_k).$$

Definition 1.5. (Butnariu & Iusem, 2000, p.69)**.** Let X be a real reflexive Banach space, $g: X \to (-\infty, +\infty]$ be Legendre, and Ω be a nonempty, convex, and closed subset of $\mathrm{int}(\mathrm{dom} g)$. The *Bregman projection* of $u \in \mathrm{int}(\mathrm{dom} g)$ onto Ω is the unique vector $P_\Omega^g(u) \in \Omega$ satisfying $D_g(P_\Omega^g(u), u) = \inf\{D_g(v, u) : v \in \Omega\}$.

Definition 1.6. (Resmerita, 2004, p.1)**.** Let X be a real reflexive Banach space and $g: X \to (-\infty, +\infty]$ be Gâteaux differentiable. Then

(1) g is called *totally convex at* $u \in \mathrm{int}(\mathrm{dom} g)$ if any $\varepsilon > 0$, we have

$v_g(u, \varepsilon) = \inf\{D_g(v, u) : y \in \mathrm{dom}\, g, \| v - u \| = \varepsilon\} > 0.$

(2) g is called *totally convex* if g is totally convex at every point $u \in \mathrm{int}(\mathrm{dom} f)$.

(3) g is called *totally convex on bounded subsets of* X if any nonempty bounded subset E of X and $t > 0$, we have $v_g(E, \varepsilon) = \inf\{v_g(u, \varepsilon) : u \in E \cap \mathrm{dom}\, g\} > 0.$

Proposition 1.7. (Resmerita, 2004, Proposition 2.2). *Let X be a real reflexive Banach space, and $g: X \to \mathbb{R}$ be Gâteaux differentiable. Then g is totally convex at $u \in X$ if and only if for any sequence $\{v_n\} \subset X$ such that $\lim_{n\to\infty} D_g(v_n, u) = 0$, we have $\lim_{n\to\infty} \| v_n - u \| = 0.$*

Proposition 1.8. (Butnariu & Iusem, 2000, Lemma 2.1.2). *Let X be a real reflexive Banach space, and $g: X \to \mathbb{R}$ be convex and Gâteaux differentiable. Then g is totally convex on bounded sets if and only if for any sequence $\{u_n\}, \{v_n\} \subset X$ such that $\{u_n\}$ is bounded and $\lim_{n\to\infty} D_g(v_n, u_n) = 0$, we have $\lim_{n\to\infty} \| v_n - u_n \| = 0.$*

Proposition 1.9. (Butnariu & Resmerita, 2006, Corollary 4.4). *Let X be a real reflexive Banach space, $g: X \to (-\infty, +\infty]$ be a Gâteaux differentiable function and totally convex on $\mathrm{int}(\mathrm{dom} g)$, Ω be a nonempty, closed, and convex subset and $u \in \mathrm{int}(\mathrm{dom} g)$. Then*

(1) $w = P_\Omega^g(u)$ if and only if $\langle \nabla g(u) - \nabla g(w), w - v \rangle \geq 0$ for all $v \in \Omega$.

(2) $D_g(v, P_\Omega^g(u)) + D_g(P_\Omega^g(u), u) \leq D_g(v, u)$ for all $v \in \Omega$.

Proposition 1.10. *Let X be a real reflexive Banach space and $g: X \to \mathbb{R}$ be a function.*

(1) [Reich & Sabach, 2010, Lemma 1]. *If g is Gâteaux differentiable and totally convex on X, $u \in X$ and $\{u_n\} \subset X$ satisfying $\{D_g(u_n, u)\}$ is bounded, then the sequence $\{u_n\}$ is bounded.*

(2) [Sabach, 2011, Proposition 2.3]. *If g is Legendre such that ∇g^* is bounded on bounded subsets, $u \in X$ and $\{u_n\} \subset X$ satisfying $\{D_g(u, u_n)\}$ is bounded, then the sequence $\{u_n\}$ is bounded.*

Definition 1.11. (Zalinescu, 2002, p.203, p.207, p.221). Let X be a Banach space. We denote by $S_1 = \{u \in X : ||u|| < 1\}$ and $B_\varepsilon = \{u \in X : ||u|| \leq \varepsilon\}$ for some $\varepsilon > 0$. Then

(1) $g : X \to \mathbb{R}$ is called *uniformly convex on bounded subsets* if $\rho_\varepsilon(\lambda) > 0$ for all $\lambda, \varepsilon > 0,$ where the function $\rho_\varepsilon : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_\varepsilon(\lambda) = \inf_{u,v \in B_\varepsilon, ||u-v|| = \lambda, \delta \in (0,1)} \frac{\delta g(u) + (1-\delta) g(v) - g(\delta u + (1-\delta) v)}{\delta(1-\delta)}.$$

(2) $g : X \to \mathbb{R}$ is called *uniformly smooth on bounded subsets* if $\lim_{\lambda \to 0} \frac{\sigma_\varepsilon(\lambda)}{\lambda} = 0$ for all $\varepsilon > 0,$ where the function $\sigma_\varepsilon : [0, \infty) \to [0, \infty)$ is defined by

$$\sigma_\varepsilon(\lambda) = \sup_{u \in B_\varepsilon, v \in S_1, \delta \in (0,1)} \frac{\delta g(u + (1-\delta)\lambda v) + (1-\delta) g(u - \delta \lambda v) - g(u)}{\delta(1-\delta)}.$$

Remark 1.12. (Naraghirad & Yao, 2013, p.7). *The function g is uniformly convex on bounded subsets if and only if g is totally convex on bounded subsets.*

Definition 1.13. (Kohsaka and Takahashi, 2005, p.509). Let X be a Banach space. Then $g : X \to (-\infty, +\infty]$ is called *strongly coercive* if $\lim_{||u|| \to +\infty} \frac{g(u)}{||u||} = +\infty.$

Proposition 1.14. [Zalinescu, 2002, Proposition 3.6.3]. *Let X be a real reflexive Banach space, $g : X \to \mathbb{R}$ be strongly coercive, continuous, and convex. Then g is bounded on bounded subsets and uniformly smooth on bounded subsets if and only if $\mathrm{dom}(g^*) = X^*,$ g^* is strongly coercive and uniformly convex on bounded subsets.*

Proposition 1.15. (Zalinescu, 2002, Proposition 3.6.4). *Let X be a real reflexive Banach space, $g : X \to \mathbb{R}$ be convex, continuous, and bounded on bounded subsets of X. Then the following statements are equivalent.*

(1) g is uniformly convex on bounded subsets and strongly coercive.

(2) $\mathrm{Dom}(g^*) = X^*,$ g^* is bounded and uniformly smooth on bounded subsets.

(3) $\mathrm{Dom}(g^*) = X^*,$ g^* is Fréchet differentiable and ∇g^* is uniformly continuous on bounded subsets.

Lemma 1.16. (Naraghirad & Yao, 2013, Lemma 2.2). *Let X be a Banach space, $r > 0$ and $g : X \to \mathbb{R}$ be convex and uniformly convex on bounded subsets. Then*

$$g(\sum_{n=1}^{m} a_n u_n) \leq \sum_{n=1}^{m} a g(u_n) - a_i a_j \rho_\varepsilon(|| u_i - u_j ||)$$

with $i, j \in \{1, 2, \ldots, m\}$, $u_n \in B_\varepsilon = \{u \in X : || u || \leq \varepsilon\}$ *and* $a_n \in [0,1]$ *such that* $\sum_{n=1}^{m} a_n = 1$, *and the function* ρ_ε *is defined as in Definition 1.11.*

We denote by $F(S) = \{w \in \Omega : Sw = w\}$ the set of fixed points of $S : \Omega \to \Omega$.

Definition 1.17. (Chang *et al.*, 2014, Definition 2.10). Let X be a reflexive Banach space, Ω be a nonempty subset of X, $S : \Omega \to \Omega$ be a mapping, and D_g be the Bregman distance. Then

(1) S is called a *Bregman quasi-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exists a real sequence $\{\delta_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \delta_n = 1$ such that

$$D_g(u, S^n v) \leq \delta_n D_g(u, v) \text{ for all } v \in \Omega \text{ and } u \in F(S).$$

(2) S is called a *Bregman totally quasi-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exist nonnegative real sequences $\{\alpha_n\}, \{\beta_n\}$ with $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$D_g(u, S^n v) \leq D_g(u, v) + \alpha_n \zeta(D_g(u, v)) + \beta_n \text{ for all } v \in \Omega \text{ and } u \in F(S).$$

(3) S is called a *Bregman firmly nonexpansive mapping* if

$$\langle \nabla g(Su) - \nabla g(Sv), Su - Sv \rangle \leq \langle \nabla g(u) - \nabla g(v), Su - Sv \rangle \text{ for all } u, v \in \Omega.$$

(4) S is called a *Bregman quasi-nonexpansive mapping* if $F(S) \neq \varnothing$ and

$$D_g(u, Sv) \leq D_g(u, v) \text{ for all } v \in \Omega \text{ and } u \in F(S).$$

Remark 1.18. [Chang *et al.*, 2014, p.42].

(1) If S *is a Bregman quasi-asymptotically nonexpansive mapping, then* S *is a Bregman totally quasi-asymptotically nonexpansive mapping with* $\zeta(\lambda) = \lambda$ *for all* $\lambda \geq 0$, $\alpha_n = \delta_n - 1$ *with* $\delta_n \geq 1$ *satisfying* $\lim_{n\to\infty} \delta_n = 1$ *and* $\beta_n = 0$; *but the converse is not true.*

(2) If S *is a Bregman firmly nonexpansive mapping, then* S *is a Bregman quasi-nonexpansive mapping.*

Definition 1.19. (Zhu & Huang, 2016, Definition 2.10)**.** Let X be a Banach space, Ω be a nonempty subset of X, $S : \Omega \to \Omega$ be a mapping. Then

(1) S is called *closed* if any sequence $\{u_n\}$ in Ω such that $\lim_{n\to\infty} u_n = u \in \Omega$ and $\lim_{n\to\infty} Su_n = v \in \Omega$, we have $Su = v$.

(2) S is called *uniformly asymptotically regular* on Ω if for all bounded subset U of Ω we have $\lim_{n\to\infty} \sup_{x \in U} || S^{n+1}u - S^n u || = 0$.

Lemma 1.20. (Chang et al., 2014, Lemma 2.16). *Let X be a real reflexive Banach space, Ω be a nonempty, closed, and convex subset of X, $g: X \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of X, $S: \Omega \to \Omega$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping. Then $F(S)$ is convex and closed.*

In order to solve (GMEP), we suppose that f satisfies the following hypotheses:

(C1) $f(u,u) = 0$ for all $u \in \Omega$.

(C2) $f(u,v) + f(v,u) \le 0$ for all $u, v \in \Omega$.

(C3) $\limsup_{\lambda \to 0} f(\lambda w + (1-\lambda)u, v) \le f(u,v)$ for all $u, v, w \in \Omega$,

(C4) For each $u \in \Omega$, $v \mapsto f(u,v)$ is convex and lower semi-continuous.

Definition 1.21. [Darvish, 2016, Definition 2.4]. Let X be a real reflexive Banach space, Ω be a nonempty, convex, and closed subset of X. Suppose that $f: \Omega \times \Omega \to \mathbb{R}$ satisfies (C1)-(C4), $\varphi: \Omega \to \mathbb{R}$ is convex and lower semi-continuous, $\psi: \Omega \to X^*$ is continuous monotone. *The mixed resolvent of f* is the mapping $\mathrm{Res}^g_{f,\varphi,\psi}: X \to 2^\Omega$ which is defined by

$$\mathrm{Res}^g_{f,\varphi,\psi}(u) = \{w \in \Omega : f(w,v) + \varphi(v) + \langle \psi(u), v - w\rangle + \langle \nabla f(w) - \nabla f(u), v - w\rangle \ge \varphi(w), \forall v \in \Omega\}.$$

Note that if $g: X \to (-\infty, +\infty]$ is strongly coercive and Gâteaux differentiable, then $\mathrm{dom}(\mathrm{Res}^g_{f,\varphi,\psi}) = X$, see [Darvish, 2016, Lemma 2.7]. We find that the formula of the function $\mathrm{Res}^g_{f,\varphi,\psi}$ contains the term $\psi(u)$ for all $u \in X$. Since $\mathrm{dom}\psi = \Omega \subset X$, the value $\psi(u)$ does not exist for all $u \in X \setminus \Omega$. Motivated by this confusion, we revise the formula of the function $\mathrm{Res}^g_{f,\varphi,\psi}$ by replacing the term $\psi(u)$, $u \in X$ by $\psi(w)$, $w \in \Omega$. This formula has been stated in (Ni & Wen, 2018, Lemma 2.5), where $\mathrm{Res}^g_{f,\varphi,\psi}$ is denoted by T^G_r as follows.

$$\mathrm{Res}^g_{f,\varphi,\psi}(u) = \{w \in \Omega : f(w,v) + \varphi(v) + \langle \psi(w), v - w\rangle + \langle \nabla f(w) - \nabla f(u), v - w\rangle \ge \varphi(w), \forall v \in \Omega\}. \quad (1.2)$$

The following lemma presents some properties of $\mathrm{Res}^g_{f,\varphi,\psi}$ which is defined by (1.2).

Lemma 1.22. (Ni & Wen, 2018, Lemma 2.5). *Let X be a real reflexive Banach space, Ω be a nonempty, closed, and convex subset of X, $g: X \to \mathbb{R}$ be Legendre and $f: \Omega \times \Omega \to \mathbb{R}$ be a bifunctional satisfying (C1)-(C4). Then*

(1) $\mathrm{Res}^g_{f,\varphi,\psi}$ is a single-valued and Bregman firmly nonexpansive mapping.

(2) $F(\mathrm{Res}^g_{f,\varphi,\psi}) = GMEP(f,\varphi,\psi)$, $GMEP(f,\varphi,\psi)$ is convex and closed.

(3) For all $u \in X$ and $v \in F(\mathrm{Res}^g_{f,\varphi,\psi})$, we have

$$D_g(v, \mathrm{Res}^g_{f,\varphi,\psi}(u)) + D_g(\mathrm{Res}^g_{f,\varphi,\psi}(u), u) \le D_g(v,u).$$

2. Main results

The result shows the strong convergence of a hybrid iteration process for a generalized mixed equilibrium problem and a Bregman totally quasi-asymptotically nonexpansive mapping in reflexive Banach spaces.

Theorem 2.1. *Let X be a real reflexive Banach space, Ω be a nonempty, closed and convex subset of X, $g : X \to \mathbb{R}$ be Legendre, strongly coercive, bounded, totally convex, and Fréchet differentiable on bounded subsets. Suppose that $f : \Omega \times \Omega \to \mathbb{R}$ satisfies (C1)-(C4), $\varphi : \Omega \to \mathbb{R}$ is lower semi-continuous and convex, $\psi : \Omega \to X^*$ is continuous monotone, $S : \Omega \to \Omega$ is a closed, uniformly asymptotically regular, and Bregman totally quasi-asymptotically nonexpansive mapping with $\{\alpha_n\}, \{\beta_n\} \subset [0,\infty)$* satisfying $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that $\mathcal{F} = F(S) \cap GMEP(f,\varphi,\psi)$ is bounded and nonempty. Let $\{z_n\}$ be a sequence generated by: $z_1 \in \Omega, \Omega_1 = \Omega$ and*

$$\begin{cases} u_n = \nabla g^*(a_n \nabla g(z_n) + (1-a_n)\nabla g(S^n z_n)) \\ v_n \in \Omega : f(v_n, v) + \varphi(v) + \langle \psi(v_n), v - v_n\rangle + \langle \nabla g(v_n) - \nabla g(z_n), v - v_n\rangle \ge \varphi(v_n), \forall v \in \Omega \\ w_n = \nabla g^*(b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n)) \\ \Omega_{n+1} = \{u \in \Omega_n : D_g(u, w_n) \le D_g(u, z_n) + \gamma_n\} \\ z_{n+1} = P^g_{\Omega_{n+1}}(z_1), n \in \mathbb{N}^* \end{cases} \tag{2.1}$$

where $\gamma_n = \alpha_n \sup\{\zeta(D_g(u, z_n)) : u \in \mathcal{F}\} + \beta_n$ and $\{a_n\}, \{b_n\} \subset [0,1]$ such that $\lim_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} b_n(1-b_n) > 0.$ Then the sequence $\{z_n\}$ strongly converges to $p = P^g_{\mathcal{F}}(z_1).$

Proof. We divide the proof of this theorem into six steps.

Step 1. *We show that $P^g_{\mathcal{F}}(x_1)$ is well-defined.* Indeed, it follows from Lemma 1.20 and Lemma 1.22 that $F(S)$ and $GMEP(f,\varphi,\psi)$ are closed and convex. Therefore, by combining this with the assumption, we obtain that $\mathcal{F} = F(S) \cap \mathrm{GMEP}(f,\varphi,\psi)$ is a nonempty, closed, and convex subset of $\Omega.$ This fact ensures that $P^g_{\mathcal{F}}(z_1)$ is well-defined.

Step 2. *We show that $P^g_{\Omega_{n+1}}(z_1)$ is well-defined.* We first claim by mathematical induction that Ω_n is convex and closed for all $n \in \mathbb{N}^*$. Obviously, for $n = 1,$ we have $\Omega_1 = \Omega$ is closed and convex. Now we suppose that Ω_k is convex and closed for some $k \in \mathbb{N}^*.$ Then, by the definition of $\Omega_{n+1},$ we have

$$\Omega_{k+1} = \{u \in \Omega_k : \langle \nabla g(z_k), u - z_k \rangle - \langle \nabla g(w_k), u - w_k \rangle \le \gamma_k - g(z_k) + g(w_k)\}. \tag{2.2}$$

By combining (2.2) with the continuity of $\nabla g(.)$, we get that Ω_{k+1} is convex and closed. Therefore, Ω_n is convex and closed for all $n \in \mathbb{N}^*$. Next, we will claim by mathematical induction that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}^*$. Obviously, we have $\mathcal{F} \subset \Omega = \Omega_1$. Now, we suppose that $\mathcal{F} \subset \Omega_k$ for some $k \in \mathbb{N}^*$. We will show that $\mathcal{F} \subset \Omega_{k+1}$. Indeed, for any $u \in \mathcal{F}$, we get $u \in \Omega_k$. By using Remark 1.4.(2), we have

$$D_g(u, u_k) = D_g(u, \nabla g^*(a_k \nabla g(z_k) + (1 - a_k)\nabla g(S^k z_k))) \le a_k D_g(u, z_k) + (1 - a_k) D_g(u, S^k z_k)$$
$$\le a_k D_g(u, z_k) + (1 - a_k)[D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k]$$
$$= D_g(u, z_k) + (1 - a_k)[\alpha_k \zeta(D_g(u, z_k)) + \beta_k] \le D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k. \tag{2.3}$$

Furthermore, by the definition of v_n and definition 1.21, we have $v_n = \mathrm{Res}^g_{f,\varphi,\psi}(z_n)$. It follows from Remark 1.18 and Lemma 1.22 that $\mathrm{Res}^g_{f,\varphi,\psi}$ is a Bregman quasi-nonexpansive mapping. Therefore $D_g(u, v_k) = D_g(u, \mathrm{Res}^g_{f,\varphi,\psi}(z_k)) \le D_g(u, z_k)$. (2.4)

Next, by using Remark 1.4. (2), we obtain

$$D_g(u, w_k) = D_g(u, \nabla g^*(b_n \nabla g(u_k) + (1 - b_k)\nabla g(S^k v_k))) \le b_k D_g(u, u_k) + (1 - b_k) D_g(u, T^k v_k)$$
$$\le b_k D_g(u, u_k) + (1 - b_k)[D_g(u, v_k) + \alpha_k \zeta(D_g(u, v_k)) + \beta_k]. \tag{2.5}$$

It follows from (2.4) and the strictly increasing property of ζ that $\zeta(D_g(u, v_k)) < \zeta(D_g(u, z_k))$. Then, from (2.4), (2.5) becomes

$$D_g(u, w_k) \le b_k D_g(u, u_k) + (1 - b_k)[D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k]. \tag{2.6}$$

By substituting (2.3) into (2.6), we have

$$D_g(u, w_k) \le b_k[D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k] + (1 - b_k)[D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k]$$
$$= D_g(u, z_k) + \alpha_k \zeta(D_g(u, z_k)) + \beta_k \le D_g(u, z_k) + \gamma_k. \tag{2.7}$$

This implies that $u \in \Omega_{k+1}$ and hence $\mathcal{F} \subset \Omega_{k+1}$. Therefore, we conclude that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}^*$. By the assumption $\mathcal{F} \ne \varnothing$, we obtain $\Omega_{n+1} \ne \varnothing$. Therefore, we find that $P^g_{\Omega_{n+1}}(z_1)$ is well-defined.

Step 3. *We show that* $\{D_g(z_n, z_1)\}, \{z_n\}$ *is bounded and* $\lim_{n\to\infty} D_g(z_n, z_1)$ *exists.* Indeed, since $z_n = P^g_{\Omega_n}(z_1)$, by Proposition 1.9, we get $D_g(y, z_n) + D_g(z_n, z_1) \le D_g(y, z_1), \forall y \in \Omega_n$. (2.8)

Let $u \in \mathcal{F}$. Since $\mathcal{F} \subset \Omega_n$, we get $u \in \Omega_n$. By choosing $y = u$ in (2.8), we obtain

$$D_g(u, z_n) + D_g(z_n, z_1) \le D_g(u, z_1). \tag{2.9}$$

This implies that $D_g(z_n, z_1) \le D_g(u, z_1) - D_g(u, z_n) \le D_g(u, z_1)$. Therefore, $\{D_g(z_n, z_1)\}$ is bounded. Then, by Proposition 1.10(1), we conclude that the sequence $\{z_n\}$ is bounded. Furthermore, we have $z_{n+1} = P^g_{\Omega_{n+1}}(z_1) \in \Omega_{n+1} \subset \Omega_n$. By choosing $y = z_{n+1}$ in (2.8), we get $D_g(z_{n+1}, z_n) + D_g(z_n, z_1) \le D_g(z_{n+1}, z_1)$. This implies that $D_g(z_n, z_1) \le D_g(z_{n+1}, z_1)$. This proves that $\{D_g(z_n, z_1)\}$ is a nondecreasing sequence. By combining this with the boundedness of the sequence $\{D_g(z_n, z_1)\}$, we conclude that the limit $\lim_{n\to\infty} D_g(z_n, z_1)$ exits.

Step 4. *We show that* $\lim_{n\to\infty} z_n = p \in \Omega$ *and* $\lim_{n\to\infty} \| z_{n+1} - z_n \| = 0$. Indeed, for $m > n$, we have $z_m = P^g_{\Omega_m}(z_1) \in \Omega_m \subset \Omega_n$. By choosing $y = z_m$ in (2.8), we get

$$D_g(z_m, z_n) + D_g(z_n, z_1) \le D_g(z_m, z_1).$$

This implies that $0 \le D_g(z_m, z_n) \le D_g(z_m, z_1) - D_g(z_n, z_1)$. (2.10)

Taking the limit (2.10) as $m, n \to \infty$ and using the existence of $\lim_{n\to\infty} D_g(z_n, z_1)$, we get

$$\lim_{n,m\to\infty} D_g(z_m, z_n) = 0. \quad (2.11)$$

By combining (2.11) with the boundedness of the sequence $\{z_n\}$, by Proposition 1.8, we have $\lim_{n,m\to\infty} \| z_n - z_m \| = 0$. (2.12)

This proves that $\{z_n\}$ is a Cauchy sequence in Ω. Since X is a Banach space and Ω is a closed subset of X, there exists $p \in \Omega$ such that $\lim_{n\to\infty} z_n = p$. Moreover, by choosing $m = n + 1$ in (2.11) and (2.12), we obtain $\lim_{n\to\infty} D_g(z_{n+1}, z_n) = 0$ (2.13)

and $\lim_{n\to\infty} \| z_{n+1} - z_n \| = 0$. (2.14)

Step 5. *We show that* $p \in \mathcal{F}$. Indeed, since $z_{n+1} = P^g_{\Omega_{n+1}}(z_1) \in \Omega_{n+1} \subset \Omega_n$, we have

$$D_g(z_{n+1}, w_n) \le D_g(z_{n+1}, z_n) + \gamma_n. \quad (2.15)$$

It follows from (2.9) and the boundedness of $\{D_g(z_n, z_1)\}$ that $\{D_g(u, z_n)\}$ is bounded for any $u \in \mathcal{F}$. Then, by using $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we find that $\lim_{n\to\infty} \gamma_n = 0$. (2.16)

Therefore, from (2.13), (2.15), and (2.16), we conclude that $\lim_{n\to\infty} D_g(z_{n+1}, w_n) = 0$. (2.17)

Let $u \in \mathcal{F}$. By (2.9) and the boundedness of $\{D_g(z_n, z_1)\}$, we obtain that $\{D_g(u, z_n)\}$ is bounded. By combining this with (2.7), we conclude that $\{D_g(u, w_n)\}$ is bounded. Furthermore, by Proposition 1.15, we find that g^* is bounded on bounded sets. Then ∇g^* is bounded on bounded sets. It follows from Proposition 1.10(2) that $\{w_n\}$ is bounded. By combining this with (2.17), from Proposition 1.8, we have $\lim_{n\to\infty} \| z_{n+1} - w_n \| = 0.$ (2.18)

It follows from (2.14) and (2.18) that $\lim_{n\to\infty} \| z_n - w_n \| = 0.$ (2.19)

Since g is uniformly Fréchet differentiable, g is uniformly continuous. Then, from (2.19) we get $\lim_{n\to\infty} \| g(z_n) - g(w_n) \| = 0.$ (2.20)

Since g is uniformly Fréchet differentiable, ∇g is uniformly continuous on bounded subsets of $X.$ Therefore, from (2.19), we have $\lim_{n\to\infty} \| \nabla g(z_n) - \nabla g(w_n) \| = 0.$ (2.21)

For any $u \in \mathcal{F},$ by using similar arguments as in the proofs of (2.3) and (2.4), we obtain

$$D_g(u,u_n) \le D_g(u,z_n) + \alpha_n \zeta(D_g(u,z_n)) + \beta_n \quad (2.22)$$

and $D_g(u,v_n) \le D_g(u,z_n).$ (2.23)

By combining (2.22) with the boundedness of $\{D_g(u,x_n)\},$ we get that $\{D_g(u,u_n)\}$ is bounded. By Proposition 1.10(2), we get that $\{u_n\}$ is bounded. It follows from (2.23) and the boundedness of $\{D_g(u,x_n)\}$ that $\{D_g(u,v_n)\}$ is bounded. Since $\{D_g(u,v_n)\}$ is bounded and $D_g(u,S^n v_n) \le D_g(u,v_n) + \alpha_n \zeta(D_g u, v_n) + \beta_n,$ we find that $\{D_g(u,S^n v_n)\}$ is bounded. Thus, from Proposition 1.10(2), we get that $\{S^n v_n\}$ is bounded. Since $\{u_n\},$ $\{S^n v_n\}$ are bounded and ∇g is bounded on bounded subsets of $X,$ we conclude that $\{\nabla g(u_n)\}$ and $\{\nabla g(S^n v_n)\}$ are bounded. Put $r = \sup_{n\in\mathbb{N}^*} \max\{\| \nabla g(u_n) \|, \| \nabla g(S^n v_n) \|\}.$ Therefore, $\nabla g(u_n), \nabla g(S^n v_n) \in B_\varepsilon = \{u \in X^* : \| u \| \le \varepsilon\}.$ By Proposition 1.14, we find that g^* is uniformly convex on bounded subsets of $X^*.$ Therefore, by Lemma 1.16, we have

$$\begin{aligned} &g^*(b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n)) \\ &\le b_n g^*(\nabla g(u_n)) + (1-b_n) g^*(\nabla g(S^n v_n)) - b_n(1-b_n)\rho_\varepsilon(\| \nabla g(u_n) - \nabla g(S^n v_n) \|), \end{aligned}$$

where ρ_ε is defined as in Definition 1.11. By using Remark 1.4.(1) and the definition of $V_f,$ we get

$$\begin{aligned} D_g(u,w_n) &= D_g(u, \nabla g^*(b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n))) = V_g(u, b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n)) \\ &= g(u) - \langle b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n), u\rangle + g^*(b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n)) \\ &= g(u) - \langle b_n \nabla g(u_n) + (1-b_n)\nabla g(S^n v_n), u\rangle \\ &\quad + b_n g^*(\nabla g(u_n)) + (1-b_n) g^*(\nabla g(S^n v_n)) - b_n(1-b_n)\rho_\varepsilon(\| \nabla g(u_n) - \nabla g(S^n v_n) \|) \\ &= b_n[g(u) - \langle \nabla g(u_n), u\rangle + g^*(\nabla g(u_n))] + (1-b_n)[g(u) - \langle \nabla g(S^n v_n), u\rangle + g^*(\nabla g(S^n v_n))] \end{aligned}$$

$$-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)$$
$$=b_nV_g(u,\nabla g(u_n))+(1-b_n)V_g(u,\nabla g(S^n v_n))-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)$$
$$=b_nD_g(u,\nabla g^*(\nabla g(u_n)))+(1-b_n)D_g(u,\nabla g^*(\nabla g(S^n v_n)))$$
$$-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)$$
$$=b_nD_g(u,u_n)+(1-b_n)D_g(u,S^n v_n)-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)$$
$$\le b_nD_g(u,u_n)+(1-b_n)[D_g(u,v_n)+\alpha_n\zeta(D_g(u,v_n))+\beta_n]$$
$$-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|). \quad (2.24)$$

Thus, by combining (2.23), (2.24) and the the strictly increasing property of ζ, we get

$$D_g(u,w_n)\le b_nD_g(u,u_n)+(1-b_n)[D_g(u,z_n)+\alpha_n\zeta(D_g(u,z_n))+\beta_n]$$
$$-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|). \quad (2.25)$$

By (2.22) and (2.25), we get $D_g(u,w_n)\le D_g(u,z_n)+\gamma_n-b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)$.

This implies that $b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)\le D_g(u,z_n)-D_g(u,w_n)+\gamma_n$. (2.26)

Furthermore, by the property of the function D_g, we have

$$|D_g(u,z_n)-D_g(u,w_n)|=|-D(z_n,w_n)+\langle\nabla g(w_n)-\nabla g(z_n),u-z_n\rangle|$$
$$\le|g(z_n)-g(w_n)|+\|\nabla g(w_n)\|.\|z_n-w_n\|+\|u-z_n\|.\|\nabla g(w_n)-\nabla g(z_n)\|. \quad (2.27)$$

Then from (2.19), (2.20), (2.21), and (2.27), we get $\lim_{n\to\infty}|D_g(u,z_n)-D_g(u,w_n)|=0$. (2.28)

By (2.16), (2.26), and (2.28) that $\lim_{n\to\infty}b_n(1-b_n)\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)=0$. (2.29)

By (2.29) and $\liminf_{n\to\infty}b_n(1-b_n)>0$, we have $\lim_{n\to\infty}\rho_\varepsilon(\|\nabla g(u_n)-\nabla g(S^n v_n)\|)=0$. (2.30)

By combining (2.30) and the property of ρ_ε, we get $\lim_{n\to\infty}\|\nabla g(u_n)-\nabla g(S^n v_n)\|=0$. (2.31)

It follows from the assumptions of g and Proposition 1.15 that ∇g^* is uniformly continuous on bounded subsets. Thus, by (2.31), we get $\lim_{n\to\infty}\|u_n-S^n v_n\|=0$. (2.32)

Furthermore, by $\nabla g=(\nabla g^*)^{-1}$ and the definition of u_n, we obtain

$\nabla g(u_n)=\nabla g(\nabla g^*(a_n\nabla g(z_n)+(1-a_n)\nabla g(S^n z_n)))=a_n\nabla g(z_n)+(1-a_n)\nabla g(S^n z_n)$. This leads to $\|\nabla g(u_n)-\nabla g(z_n)\|=(1-a_n)\|\nabla g(S^n z_n)-\nabla g(z_n)\|$. (2.33)

By $\lim_{n\to\infty}a_n=1$, the boundedness of $\{z_n\}$, (2.33), we get $\lim_{n\to\infty}|\nabla g(u_n)-\nabla g(z_n)\|=0$. (2.34)

Since ∇g^* is uniformly continuous on bounded subsets, from (2.34), we have

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \quad (2.35)$$

By combining (2.32) and (2.35), we obtain $\lim_{n\to\infty}\|z_n-S^n v_n\|=0$. (2.36)

It follows from (2.36) and $\lim_{n\to\infty} z_n = p$ that $\lim_{n\to\infty} S^n v_n = p.$ Thus, by combining this with the asymptotically regular property of S and $\| S^{n+1}v_n - p \| \le \| S^{n+1}v_n - S^n v_n \| + \| S^n v_n - p \|,$ we conclude that $\lim_{n\to\infty} S^{n+1}v_n = p.$ This leads to $\lim_{n\to\infty} S(S^n v_n) = \lim_{n\to\infty} S^{n+1} v_n = p.$ Since S is closed, we conclude that $Sp = p$ and hence $p \in F(S).$

Next, we will prove that $p \in GMEP(f, \varphi, \psi)$. Since $v_n = \mathrm{Res}^g_{f,\varphi,\psi}(z_n),$ we obtain

$$f(v_n, v) + \varphi(v) + \langle \psi(v_n), v - v_n \rangle + \langle \nabla g(v_n) - \nabla g(z_n), v - v_n \rangle \ge \varphi(v_n) \text{ for all } v \in \Omega. \quad (2.37)$$

Then, from the condition (C_2) and (2.37), we have

$$f(v, v_n) \le -f(v_n, v) \le \langle \psi(v_n), v - v_n \rangle + \langle \nabla g(v_n) - \nabla g(z_n), v - v_n \rangle + \varphi(v) - \varphi(v_n). \quad (2.38)$$

Furthermore, since g and ∇g are uniformly continuous on bounded subsets of $X,$ by (2.36), we get $\lim_{n\to\infty} \| g(z_n) - g(S^n v_n) \| = \lim_{n\to\infty} \| \nabla g(z_n) - \nabla g(S^n v_n) \| = 0.$ (2.39)

We have $| D_g(u, z_n) - D_g(u, S^n v_n) | = | -D(z_n, S^n v_n) + \langle \nabla g(S^n v_n) - \nabla g(z_n), u - z_n \rangle |$

$$\le | g(z_n) - g(S^n v_n) | + \| \nabla g(S^n v_n) \| . \| z_n - S^n v_n \| + \| u - z_n \| . \| \nabla g(S^n v_n) - \nabla g(z_n) \| . \quad (2.40)$$

By (2.36), (2.39) and (2.40), we get that $\lim_{n\to\infty} | D_g(u, z_n) - D_g(u, S^n v_n) | = 0.$ (2.41)

For $u \in \mathcal{F},$ by Lemma 1.22 and $v_n = \mathrm{Res}^g_{f,\varphi,\psi}(z_n),$ we find that

$$D_g(v_n, z_n) \le D_g(u, z_n) - D_g(u, v_n) \le D_g(u, z_n) - D_g(u, S^n v_n) + \alpha_n \zeta(D_g(u, v_n)) + \beta_n. \quad (2.42)$$

It follows from (2.41), (2.42) and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ that $\lim_{n\to\infty} D_g(v_n, z_n) = 0.$ Since $\{z_n\}$ is bounded, by Proposition 1.8, we have $\lim_{n\to\infty} \| v_n - z_n \| = 0.$ Since ∇g is uniformly continuous on bounded subsets, we get $\lim_{n\to\infty} \| \nabla g(z_n) - \nabla g(v_n) \| = 0.$ Therefore, by using (2.38), the lower semi-continuous property of $\varphi,$ the lower semi-continuous property in the second variable of f and the continuous property of $\psi,$ we have

$$f(v, p) \le \langle \psi(p), v - p \rangle + \varphi(v) - \varphi(p)$$

and hence $f(v, p) + \langle \psi(p), p - y \rangle + \varphi(p) - \varphi(v) \le 0$ for all $v \in \Omega.$ (2.43)

For all $t \in (0,1],$ put $v_t = tv + (1-t)p.$ Since $v, p \in \Omega$ and Ω is convex, we have $v_t \in \Omega.$ Thus, replacing v by v_t in (2.43), we get $f(v_t, p) + \langle \psi(p), p - v_t \rangle + \varphi(p) - \varphi(v_t) \le 0.$ (2.44)

Then, by using the condition $(C_1),$ the convexity in the second variable of $f,$ the convexity of φ and (2.44), we have

$$0 = f(v_t, v_t) = f(v_t, v_t) + \langle \psi(p), v_t - v_t \rangle + \varphi(v_t) - \varphi(v_t)$$
$$\leq tf(v_t, v) + (1-t)f(v_t, p) + t\langle \psi(p), v - v_t \rangle + (1-t)\langle \psi(p), p - v_t \rangle + t\varphi(v) + (1-t)\varphi(p) - \varphi(v_t)$$
$$= t[f(v_t, v) + \langle \psi(p), v - v_t \rangle + \varphi(v) - \varphi(v_t)] + (1-t)[f(v_t, p) + \langle \psi(p), p - v_t \rangle + \varphi(p) - \varphi(v_t)]$$
$$\leq t[f(v_t, v) + \langle \psi(p), y - v_t \rangle + \varphi(v) - \varphi(v_t)].$$

This leads to $f(v_t, v) + \langle \psi(p), v - v_t \rangle + \varphi(v) - \varphi(v_t) \geq 0$ by $t \in (0,1]$. Letting $t \to 0^+$ and using the condition (C_3), we have $f(p, v) + \langle \psi(p), y - p \rangle + \varphi(v) - \varphi(p) \geq 0$. This proves that $p \in GMEP(f, \varphi, \psi)$. Therefore, $p \in \mathcal{F} = F(S) \cap GMEP(f, \varphi, \psi)$.

Step 6. *We show that* $p = P^g_{\mathcal{F}}(z_1)$. Indeed, since $z_{n+1} = P^g_{\Omega_{n+1}}(z_1)$, by Proposition 1.9, we have $\langle \nabla g(z_1) - \nabla g(z_{n+1}), z_{n+1} - v \rangle \geq 0$ for all $v \in \Omega_{n+1}$. Let $u \in \mathcal{F}$. Since $\mathcal{F} \subset \Omega_{n+1}$, we get $u \in \Omega_{n+1}$. By choosing $v = u$ in the above inequality, we get $\langle \nabla g(z_1) - \nabla g(z_{n+1}), z_{n+1} - u \rangle \geq 0$. Taking $n \to \infty$, using $\lim_{n\to\infty} z_n = p$ and the uniform continuous on bounded subsets of ∇g, we have $\langle \nabla g(z_1) - \nabla g(p), p - u \rangle \geq 0$ for all $u \in \mathcal{F}$. By Proposition 1.9, we find that $p = P^g_{\mathcal{F}}(z_1)$. □

Remark 2.2. *(1) Theorem 2.1 is an extension of Alizadeh and Moradlou (2016), Theorem 3.1] from a generalized hybrid mapping in Hilbert spaces to a Bregman totally quasi-asymptotically nonexpansive mapping, and from an equilibrium problem to a generalized mixed equilibrium problem in reflexive Banach spaces.*

(2) Since Theorem 3.1 by Alizadeh and Moradlou (2016) is an extension of Theorem 3.1 by Tada and Takahashi (2007), Theorem 2.1 is also an extension of Theorem 3.1 by Tada and Takahashi (2007),

(3) By Remark 1.8(2), we conclude that the conclusion of Theorem 2.1 holds when S *is a Bregman quasi-asymptotically nonexpansive mapping.*

Finally, an example is given to illustrate the proposed iteration.

Example 2.3. Let $X = \mathbb{R}, \Omega = [0, 0.9]$, $g(x) = u^2$ for all $x \in \mathbb{R}$, and $S(u) = u^2$, $\varphi(u) = 10u^2$, $\psi(u) = 2u$, $f(u, v) = -9u^2 + 4uv + 5v^2$ for all $u, v \in \Omega$. Then

(1) By calculating, we get $\nabla g(u) = 2u$, $g^*(w) = \dfrac{w^2}{4}$, $\nabla g^*(w) = \dfrac{w}{2}$ for all $u, w \in \mathbb{R}$.

(2) For all $u, v \in \mathbb{R}$, we have $D_g(u, v) = u^2 - v^2 - 2v(u - v) = (u - v)^2$.

(3) We have $F(S) = \{0\}$. Therefore, for $w \in F(S)$ and $u \in \Omega$, we obtain

$$D_g(w, S^n u) = (0 - S^n u)^2 = (u)^{2^{n+1}} \leq u^2 = D_g(0, u) = D_g(w, u).$$

This proves that S is a Bregman totally quasi-asymptotically nonexpansive mapping with $\alpha_n = \beta_n = 0$ for all $n \in \mathbb{N}^*$.

(4) By directly checking, we find that f satisfies the conditions (C_1) - (C_4).

(5) We find the formula of $w = \text{Res}^g_{f,\varphi,\psi}(u)$ for $u \in X, w \in \Omega$ as in (1.2). Indeed, $w = \text{Res}^g_{f,\varphi,\psi}(u)$ if $f(w,v) + \varphi(v) + \langle \psi(w), v - w\rangle + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq \varphi(w), v \in \Omega.$ (2.45)

By substituting $f, \varphi, \psi, \nabla g$ into (2.45) and by directly calculating, we get

$$15v^2 + (8w - 2u)v + 2uw - 23w^2 \geq 0.$$

Put $h(v) = 15v^2 + (8w - 2u)v + 2uw - 23w^2$ for all $v \in \Omega.$ Then $h(v)$ is a quadratic function and $\Delta = (38w - 2u)^2$. We consider the following cases.

Case 1. $\Delta > 0.$ Then the equation $h(v) = 0$ has two solutions: $v_1 = w \in \Omega$ and $v_2 = \dfrac{-23w + 2u}{15}.$ In order to $h(v) \geq 0$ for all $v \in \Omega,$ we have the following two cases:

Case 1.1. $v_1 = 0.9$ and $v_1 < v_2$. Then $w = v_1 = 0.9,$ $v_2 = \dfrac{-20.7 + 2u}{15} > 0.9,$ hence $u > 17.1.$

Case 1.2. $v_1 = 0$ and $v_2 < v_1$. Then $w = v_1 = 0$ and $v_2 = \dfrac{2u}{15} < 0.$ This leads to $u < 0.$

Case 2. $\Delta \leq 0.$ Then $w = \dfrac{u}{19}$ and $h(v) \geq 0$ for all $v \in \Omega.$ Since $w \in \Omega,$ we have $0 \leq \dfrac{u}{19} \leq 0.9$ and hence $0 \leq u \leq 17.1.$ Therefore, $\text{Res}^g_{f,\varphi,\psi}(u) = w = 0$ if $u < 0,$ $\text{Res}^g_{f,\varphi,\psi}(u) = w = \dfrac{u}{19}$ if $0 \leq u \leq 17.1$ and $\text{Res}^g_{f,\varphi,\psi}(u) = w = 0.9$ if $u > 17.1.$

By the above, all assumptions in Theorem 2.1 are satisfied with the given functions $f, \varphi, \psi, T.$ Therefore, by Theorem 2.1, the sequence $\{z_n\}$ which is defined by (2.1) converges to $0 \in \mathcal{F} = F(S) \cap GMEP(f, \varphi, \psi).$ Next, by choosing $a_n = \dfrac{n}{n+2},$ $b_n = \dfrac{n+1}{3n+2}$ for all $n \in \mathbb{N}^*,$ and $z_1 = 0.5 \in \Omega,$ we have $P^g_{\mathcal{F}}(z_1) = \{0\}.$ The sequence (2.1) becomes

$$\begin{cases} u_n = \dfrac{n}{n+2} z_n + \dfrac{2}{n+2}(z_n)^{2^n}, v_n = \dfrac{z_n}{19} \\ w_n = \dfrac{n+1}{3n+2} u_n + \dfrac{2n+1}{3n+2}(v_n)^{2^n}, z_{n+1} = \dfrac{z_n + w_n}{2}. \end{cases} \quad (2.46)$$

The convergence of iteration (2.46) is presented by the following figure.

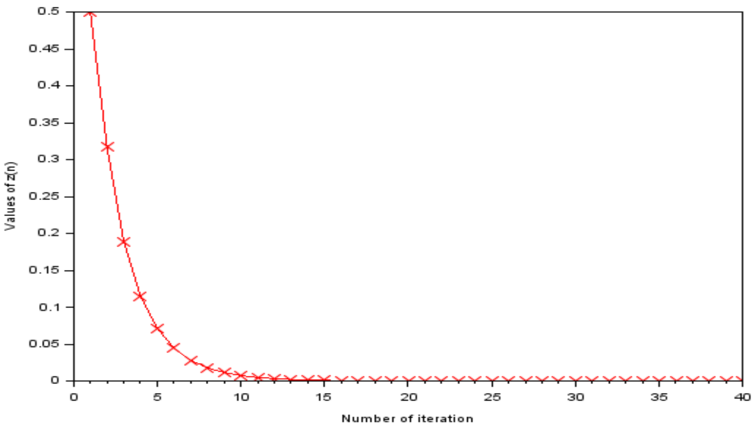

Figure 1. *The convergence of the sequence (2.46) to 0*

3. Conclusion

In this paper, a hybrid iterative method is proposed for finding common elements of the solution set of a generalized mixed equilibrium problem and the fixed point set of a Bregman totally quasi-asymptotically nonexpansive mapping. After that, a strong convergence result for the proposed iteration is proved in reflexive Banach spaces. This result is an improvement of the main results in (Alizadeh & Moradlou, 2016) and (Tada & Takahashi, 2007) from a generalized hybrid mapping, a nonexpansive mapping, and an equilibrium problem in Hilbert spaces to a Bregman totally quasi-asymptotically nonexpansive mapping and a generalized mixed equilibrium problem in reflexive Banach spaces. As an application, we obtain the convergence result for a generalized mixed equilibrium problem and a Bregman quasi-asymptotically nonexpansive mapping in reflexive Banach spaces. Moreover, we give a numerical example to illustrate the proposed iterative method.

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SỰ HỘI TỤ MẠNH CỦA DÃY LẶP LAI GHÉP CHO BÀI TOÁN CÂN BẰNG HỖN HỢP TỔNG QUÁT VÀ ÁNH XẠ TỰA TIỆM CẬN KHÔNG GIÃN HOÀN TOÀN BREGMAN TRONG KHÔNG GIAN BANACH

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TÓM TẮT

Mục đích của nghiên cứu này là kết hợp khoảng cách Bregman với phương pháp chiếu thu hẹp để giới thiệu một dãy lặp lai ghép mới cho bài toán cân bằng hỗn hợp tổng quát và ánh xạ tựa tiệm cận không giãn hoàn toàn Bregman. Sau đó, với những điều kiện thích hợp, chúng tôi chứng minh rằng dãy lặp được đề xuất hội tụ mạnh đến hình chiếu Bregman của điểm xuất phát lên giao của tập nghiệm bài toán cân bằng hỗn hợp tổng quát và tập điểm bất động của ánh xạ tựa tiệm cận không giãn hoàn toàn Bregman trong không gian Banach phản xạ. Định lí này cải tiến kết quả trong (Alizadeh & Moradlou, 2016) từ ánh xạ lai ghép tổng quát và bài toán cân bằng trong không gian Hilbert sang ánh xạ tựa tiệm cận không giãn hoàn toàn Bregman và bài toán cân bằng hỗn hợp tổng quát trong không gian Banach phản xạ. Kết quả được áp dụng cho bài toán cân bằng hỗn hợp tổng quát và ánh xạ tựa tiệm cận không giãn Bregman trong không gian Banach phản xạ. Đồng thời, một ví dụ được đưa ra để minh họa cho dãy lặp được đề xuất.

Từ khóa: ánh xạ tựa tiệm cận không giãn hoàn toàn Bregman; bài toán cân bằng hỗn hợp tổng quát; dãy lặp lai ghép; không gian Banach phản xạ