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**Research Article**

# A WEIGHTED LORENTZ ESTIMATE FOR DOUBLE-PHASE PROBLEMS

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## ABSTRACT

*Double-phase problems were modeled by minimizing the problems of a class of integral energy functionals with non-standard growth conditions. They have many applications in physics, such as nonlinear elasticity, fluid dynamics, and homogenization. The present paper provides a global gradient estimate for distribution solutions to double-phase problems in Lorentz spaces associated with a Muckenhoupt weight. In particular, this work is a weighted version of the main result found by Tran and Nguyen (2021). Our method is based on a construction of the weighted distribution inequality on fractional maximal operators, which have close relations to Riesz potential.*

***Keywords:*** distribution inequality; Double-phase problems; gradient estimates; weighted Lorentz spaces

## 1. Introduction

The calculus of variations is concerned with the minima and maxima of functionals. The search for a minimizer of a functional leads to solving the associated Euler–Lagrange equation. In recent years, researchers have been attracted by issues of the calculus of variations such as the existence of local minimizers, regularity properties of minimizers of energies, etc. This is because it has many applications for the large field of science. This paper considers the regularity properties of minimizers of a class of integral energy functionals. Specifically, we studied double-phase problems modeled by minimizing problems of a class of integral energy functionals with non-standard growth conditions. They have many applications in physics, such as nonlinear elasticity, fluid dynamics, and homogenization.

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Our intention is to build a global weighted Lorentz estimate for a non-uniformly elliptic equation which is a form of double-phase problems. The equation is given by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F\right) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset R^n$. is a bounded open domain with $n \geq 2$ and $F:\Omega \to R^n$ is a vector field. The coefficient function $a:\Omega \to [0,\infty)$ and numbers *p and q* satisfy the following assumption

$$0 \leq a(\cdot) \in C^{0,\beta},\ \beta \in (0,1];\ \text{and } 1 < p < q \leq \left(1+\frac{\beta}{n}\right)p. \tag{1.2}$$

The equation in (1.1) is regarded as the Euler-Lagrange equation of the functional

$$v \mapsto \mathcal{F}(v,\Omega) - \int_{\Omega} \left\langle |F|^{p-2}F + a(x)|F|^{q-2}F, \nabla v \right\rangle dx \ ,$$

where $\mathcal{F}(v,\Omega) := \int_{\Omega}\left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right)dx$ is called double phase functional. The functional $\mathcal{F}$ was first studied by Zhikov (Zhikov, 1986, 1995, 1997) to describe the change of ellipticity according to the positivity of the function $a$. The energy functional $\mathcal{F}$ has $p-$growth in the gradient on the set $\{a(x)=0\}$ and $q-$growth on the set $\{a(x)>0\}$.

Recently, there have been many studies on the regularity of double-phase problems associated with the Calderón–Zygmund theory, see (Baroni, & Colombo, & Mingione, 2015, 2016, 2018) và (Colombo, & Mingione, 2015a, 2015b, 2016). Colombo and Mingione (Colombo & Mingione, 2016) established the local Calderón–Zygmund estimates for equation (1.1) under sufficient conditions $a, p, q.$ The main result is given by the following

$$\left(|F|^p + a(x)|F|^q\right) \in L^{\gamma}_{loc} \ \Rightarrow\ \left(|\nabla u|^p + a(x)|\nabla u|^q\right) \in L^{\gamma}_{loc}, \tag{1.3}$$

holds for $\gamma > 1$, under assumption $\frac{q}{p} < 1+\frac{\beta}{n}$. With the case $\frac{q}{p} > 1+\frac{\beta}{n}$, Esposito, Leonetti, and Mingione (2004 ) showed that (1.3) fails to hold. Later, there have been studies that continue developing the result in (1.3). Byun and Oh (2017) extended (1.3) up to the boundary which has the condition of $\partial\Omega$ is the $C^{0,\beta^+}$ domain, $\beta^+ \in [0,1]$. De Filippis and Mingione (2020) proved that the result (1.3) still holds in the delicate limiting case $\frac{q}{p} = 1+\frac{\beta}{n}$. Furthermore, Tran and Nguyen (2021) provided the global estimates in the Lorentz spaces for the problem (1.1) according to the bounded property of fractional maximal operators.

We apply a technique called fractional maximal distribution functions (FMDs) to establish estimates for solutions to the problem (1.1). We will provide the level-set inequalities by using the property of fractional maximal operators, comparison estimate, and Vitali's covering lemma via FMDs. This method was applied to many different problems by Tran and Nguyen (2021a, 2021b, 2022a, 2022b) and Tran, Nguyen, and Nguyen, (2022). The technique FMDs was proposed based on the good-λ technique (see Tran & Nguyen, 2019a, 2019b, 2020 and Nguyen & Tran, 2020). However, it brings out a new sight as an application in regularity and Calderón-Zygmund type estimates.

In the present article, we study a class of more general equations than the equation of (1.1). The equations are given by

$$\begin{cases} \operatorname{div}\left(\mathcal{A}\left(x,\nabla u\right)\right) & = \operatorname{div}\left(\mathcal{B}\left(x,F\right)\right) \quad \text{in} \quad \Omega, \\ \qquad u & = \qquad 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset R^n$ is a bounded open domain with $n \geq 2$ and $F:\Omega \to R^n$ is a vector field. The coefficient function $a:\Omega \to [0,\infty)$ and numbers $p$ *and* $q$ satisfy conditions (1.2). The nonlinear operator $\mathcal{A}:\Omega\times R^n \to R^n$ is measurable with $x \in \Omega$, $C^1-$regular in $\zeta \in R^n$ and meets the following conditions with fixed constants $0<\nu<L<\infty$

$$\begin{cases} \left|\mathcal{A}(x,\zeta)\right| + \left|\partial_\zeta \mathcal{A}(x,\zeta)\right|\left|\zeta\right| \leq L\left(\left|\zeta\right|^{p-1} + a(x)\left|\zeta\right|^{q-1}\right); \\ \nu\left(\left|\zeta\right|^{p-2} + a(x)\left|\zeta\right|^{q-2}\right)\left|\chi\right|^2 \leq \left\langle \partial_\zeta \mathcal{A}(x,\zeta)\chi,\chi\right\rangle; \\ \left|\mathcal{A}\left(x_1,\zeta\right) - \mathcal{A}\left(x_2,\zeta\right)\right| \leq L\left|a\left(x_1\right) - a\left(x_2\right)\right|.\left|\zeta\right|^{q-1}, \end{cases} \tag{1.5}$$

for all $x, x_1, x_2 \in \Omega$ and $\zeta, \chi \in R^n \setminus \{0\}$. We remark that the condition $(1.5)_2$ implies

$$\tilde{\nu}\left[\left(\left|\zeta_1\right|^2 + \left|\zeta_2\right|^2\right)^{\frac{p-2}{2}} + a(x)\left(\left|\zeta_1\right|^2 + \left|\zeta_2\right|^2\right)^{\frac{q-2}{2}}\right]\left|\zeta_1 - \zeta_2\right|^2 \leq \left\langle \mathcal{A}\left(x,\zeta_1\right) - \mathcal{A}\left(x,\zeta_2\right), \zeta_1 - \zeta_2\right\rangle, \tag{1.6}$$

where $\tilde{\nu} = \tilde{\nu}(n,p,q,\nu)$ is a positive constant. If $2 \leq p < q$, we can write

$$\tilde{\nu}\left(\left|\zeta_1 - \zeta_2\right|^p + a(x)\left|\zeta_1 - \zeta_2\right|^q\right) \leq \left\langle \mathcal{A}\left(x,\zeta_1\right) - \mathcal{A}\left(x,\zeta_2\right), \zeta_1 - \zeta_2\right\rangle. \tag{1.7}$$

On the right-hand side, the Carathéodory vector field $\mathcal{B}:\Omega\times R^n \to R^n$ satisfies the following growth conditions

$$\left|\mathcal{B}(x,\zeta)\right| \leq L\left(\left|\zeta\right|^{p-1} + a(x)\left|\zeta\right|^{q-1}\right), \tag{1.8}$$

In the rest of the paper, we use the notation

$$\mathcal{H}(x,\zeta) = \left|\zeta\right|^p + a(x)\left|\zeta\right|^q, \tag{1.9}$$

for every $x \in \Omega, \zeta \in R^n$.

Throughout this paper, we always consider $u \in W^{1,\mathcal{H}}(\Omega)$ a distributional solution to (1.4) under assumptions (1.2) and (1.5), and $F \in L^{\mathcal{H}}(\Omega)$, where $W^{1,\mathcal{H}}(\Omega)$ and $L^{\mathcal{H}}(\Omega)$ will be introduced in Section 2. Now we present the main results of the paper in the followings.

***Theorem 1.1.*** *Let $\alpha \in [0,n)$, $\omega \in A_\infty$ and $a > 0$. Assume that $u \in W^{1,\mathcal{H}}(\Omega)$ is a distributional solution to* (1.4) *under assumptions* (1.2), (1.5), (1.8), *and $F \in L^{\mathcal{H}}(\Omega)$, one can find two positive constants $b = b(data)$ and $\varepsilon_0 = \varepsilon_0(data)$ such that the following inequality*

$$d^{\omega}_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\varepsilon^{-a}\lambda\right) \le C\varepsilon d^{\omega}_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}(\lambda) + d^{\omega}_{\mathbf{M}_\alpha(\mathcal{H}(x,F))}\left(\varepsilon^{b}\lambda\right), \tag{1.10}$$

*holds for $0 < \varepsilon < \varepsilon_0$ and $\lambda > 0$.*

***Theorem 1.2.*** *Let $\alpha \in [0,n)$, $\omega \in A_\infty$, $0 < s < \infty$ and $0 < t \le \infty$. Assume that $u \in W^{1,\mathcal{H}}(\Omega)$ is a distributional solution to* (1.4) *under assumptions* (1.2), (1.5), and (1.8) *and $F \in L^{\mathcal{H}}(\Omega)$. Then there exists $C = C(s,t,data) > 0$ such that*

$$\left\|\mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)\right\|_{L^{s,t}_\omega(\Omega)} \le C\left\|\mathbf{M}_\alpha\left(\mathcal{H}(x,F)\right)\right\|_{L^{s,t}_\omega(\Omega)}. \tag{1.11}$$

## 2. Notation and preliminaries

In this section, we will introduce some notations, definitions, and properties used throughout the paper. In what follows, $C$ stands for a general positive constant that depends on some parameters such as $n, p$ and $q$,. The accurate value of $C$ varies in different lines. With $n \ge 2$, the domain $\Omega \subset R^n$ is an open bounded set and diameter of $\Omega$ will be denoted by $\mathrm{diam}(\Omega)$. We will denote $\Omega_R(x_0) = \Omega \cap B_R(x_0)$, where $B_R(x_0) := \{\xi \in R^n : |\xi - x_0| < R\}$ is an open ball in $R^n$ with center $x_0$ and radius $R > 0$. We write $\mathcal{L}^n(A)$ for Lebesgue measure of a set $A \subset R^n$. With the coefficient function $a$, we write $[a]_{\beta;S} = \sup_{x_1,x_2 \in S; x_1 \ne x_2} \frac{|a(x_2) - a(x_1)|}{|x_2 - x_1|^\beta}$, for any $S \subset \Omega$. For simplicity of notation, we let *data* stand for the set of parameters that will affect the constant dependence in our statements below. In the sequel, we use

$$data \equiv data\left(n, q, p, \beta, \nu, L, [a]_{\beta;\Omega}, \|a\|_{L^\infty(\Omega)}, \|\mathcal{H}(x,\nabla u)\|_{L^1}, [\omega]_{\mathcal{A}_\infty}, diam(\Omega), \varepsilon_0\right).$$

Let us take the definition of Musielak-Orlicz and Musielak-Orlicz-Sobolev spaces according to the operator $\mathcal{H}$ in (1.9).

***Definition 2.1. (Musielak-Orlicz spaces)*** *Let* $k:\Omega \to R^n$ *is a measurable function, we say* $k$ *belongs to the Musielak-Orlicz class* $K^{\mathcal{H}}(\Omega)$ *if it satisfies* $\int_\Omega \mathcal{H}(x,k(x))dx < +\infty.$

*The Musielak-Orlicz spaces* $L^{\mathcal{H}}(\Omega)$ *is the is the smallest vectorial space containing* $K^{\mathcal{H}}(\Omega)$ *and norm* $\|.\|_{L^{\mathcal{H}}(\Omega)}$ *is given by*

$$\|k\|_{L^{\mathcal{H}}(\Omega)} = \inf\left\{\mu > 0: \int_\Omega \mathcal{H}\left(x, \frac{|k(x)|}{\mu}\right)dx \le 1\right\}.$$

***Definition 2.2. (Musielak-Orlicz-Sobolev spaces)*** *The Musielak-Orlicz-Sobolev space* $W^{1,\mathcal{H}}(\Omega)$ *is the set of all measurable functions* $k \in L^{\mathcal{H}}(\Omega)$ *such that* $\nabla k \in L^{\mathcal{H}}(\Omega)$. *The norm of the space* $W^{1,\mathcal{H}}(\Omega)$ *is given by*

$$\|k\|_{W^{1,\mathcal{H}}(\Omega)} = \|k\|_{L^{\mathcal{H}}(\Omega)} + \|\nabla k\|_{L^{\mathcal{H}}(\Omega)}.$$

*Furthermore, the space* $W_0^{1,\mathcal{H}}(\Omega)$ *is defined as the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,\mathcal{H}}(\Omega)$.

***Definition 2.3. (Distributional solution)*** *A function* $u \in W^{1,\mathcal{H}}(\Omega)$ *is a distributional solution to* (1.4) *under assumptions* (1.2), (1.5) *if*

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle dx = \int_\Omega \langle \mathcal{B}(x,F), \nabla\varphi\rangle dx, \tag{2.1}$$

*for every* $\varphi \in C_0^\infty(\Omega)$.

We will use an important result of the distributional solution, and its proof can be found in Proposition 3.5 (Byun & Oh, 2017).

***Lemma 2.4.*** *Let* $u \in W^{1,\mathcal{H}}(\Omega)$ *be a distributional solution to* (1.4) *under conditions* (1.2), (1.5), *and* $F \in L^{\mathcal{H}}(\Omega)$. *Then the following variational formula*

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle dx = \int_\Omega \langle \mathcal{B}(x,F), \nabla\varphi\rangle dx, \tag{2.2}$$

*holds for every test function* $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$.

Next, we will introduce the doubling property of weight that is used throughout the article as below.

***Definition 2.5. (Muckenhoupt weights)*** *Let a weight* $\omega: R^n \to R^+$ *be a locally integrable function, we say that* $\omega \in A_\infty$ *if there exist constants* $c, \delta > 0$ *satisfying*

$$\omega(A) \le c\left(\frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B)}\right)^\delta \omega(B), \tag{2.3}$$

*for every ball* $B \subset R^n$ *and all measurable subsets* $A \subset B$, *where* $\omega(A) := \int_A \omega(x)dx$. *We will set* $[\omega]_{A_\infty} = (c, \delta)$.

***Definition 2.6. (Distribution functions)*** *Let* $\omega \in A_\infty$, $K \subset R^n$ *and a measurable function* $f$ *on* $\Omega$, *and the distribution function*
$d_f^\omega(K,.)$ *is given by*

$$d_f^\omega(K,\lambda) = \int_{K \cap \{x \in \Omega : |f(x)| > \lambda\}} \omega(x)dx \text{ with } \lambda \geq 0.$$

We remark that if $\omega \equiv 1$, we write $d_f(K,\lambda) = \mathcal{L}^n(\{x \in K \cap \Omega : |f(x)| > \lambda\})$. Moreover, if $\Omega \subset \mathrm{K}$ we write $d_f^\omega(\lambda), d_f(\lambda)$ instead of $d_f^\omega(K,\lambda), d_f(K,\lambda)$ for short.

***Definition 2.7. (Weighted Lorentz spaces)*** *Let* $s \in (0,\infty)$, $t \in (0,\infty]$ *and* $\omega \in A_\infty$, *the weighted Lorentz space* $L_\omega^{s,t}(\Omega)$ *is the set of all Lebesgue measurable* $f$ *on* $\Omega$ *such that* $\|f\|_{L_\omega^{s,t}(\Omega)} < +\infty$, *where*

$$\|f\|_{L_\omega^{s,t}(\Omega)} := \left[ s\int_0^\infty \lambda^t \omega(\{x \in \Omega : |f(x)| > \lambda\})^{\frac{t}{s}} \frac{d\lambda}{\lambda} \right]^{\frac{1}{t}}, \tag{2.4}$$

*if* $t < \infty$ *and*

$$\|f\|_{L_\omega^{s,\infty}(\Omega)} := \sup_{\lambda > 0} \lambda \omega(\{x \in \Omega : |f(x)| > \lambda\})^{\frac{1}{t}}.$$

***Definition 2.8. (Fractional maximal function)*** *Let* $0 \leq \alpha \leq n$, *the maximal operator* $\mathbf{M}_\alpha$ of $f$ is given by

$$\mathbf{M}_\alpha f(\xi) = \sup_{\rho > 0} \rho^\alpha \frac{1}{\mathcal{L}^n(B_\rho(\xi))} \int_{B_\rho(\xi)} |f(y)|dy, \quad \xi \in R^n, \tag{2.5}$$

*where* $f \in L_{loc}^1(R^n)$.

***Lemma 2.9.*** (Tran & Nguyen, 2021, Lemma 2.8) *Let* $s \geq 1$ *and* $0 \leq \alpha < \frac{n}{s}$. *There exists a constant* $C = C(\alpha, n) > 0$ *such that for all* $\lambda > 0$ *there holds*

$$\mathcal{L}^n(\{x \in R^n : \mathbf{M}_\alpha f(x) > \lambda\}) \leq C\left(\frac{1}{\lambda^s}\int_{R^n} |f(y)|^s dy\right)^{\frac{n}{n-\alpha s}}, \tag{2.6}$$

*where* $f \in L^s(R^n)$.

## 3. Comparison results

In this section, we assume that $x_0 \in \Omega$ and $R > 0$. For simplicity of notation, we will use $\Omega_R = \Omega_R(x_0)$. Let us recall the following comparison estimates, which have been proved in Tran and Nguyen (2021) and Byun and Oh (2017).

***Theorem 3.1.*** (Tran & Nguyen, 2021, Lemma 3.5) *Let* $u \in W^{1,\mathcal{H}}(\Omega)$ *be a distributional solution to* (1.4) *under conditions* (1.2), (1.5), *and* $F \in L^{\mathcal{H}}(\Omega)$. *Assume that* $v \in u + W_0^{1,\mathcal{H}}(\Omega_R)$ *is the unique distribution to the following problem*

$$\begin{cases} div(\mathcal{A}(x,\nabla v)) = 0 & in \quad \Omega_R, \\ \qquad v = u & on \quad \partial\Omega_R. \end{cases} \tag{3.1}$$

*Then there exists a constant* $C = C(data) > 0$ *satisfying*

$$\frac{1}{\mathcal{L}^n(\Omega_R)}\int_{\Omega_R}\mathcal{H}(x,\nabla u-\nabla v)dx \le \frac{\varepsilon}{\mathcal{L}^n(\Omega_R)}\int_{\Omega_R}\mathcal{H}(x,\nabla u)dx + \frac{C\varepsilon^{-\eta}}{\mathcal{L}^n(\Omega_R)}\int_{\Omega_R}\mathcal{H}(x,F)dx, \tag{3.2}$$

*for every* $\varepsilon \in (0;1)$ *small enough; where* $\eta = \max\left\{0; \dfrac{2-p}{p-1}\right\}$.

***Corollary 3.2.*** *Let* $u \in W^{1,\mathcal{H}}(\Omega)$ *be a distributional solution to* (1.4) *under conditions* (1.2), (1.5), *and* $F \in L^{\mathcal{H}}(\Omega)$. *Then there exists a constant* $C = C(data) > 0$ *satisfying*

$$\int_{\Omega}\mathcal{H}(x,\nabla u)dx \le C\int_{\Omega}\mathcal{H}(x,F)dx. \tag{3.3}$$

*Proof.* Applying Theorem 3.1 to $\Omega \subset B_R(x_0)$, we may conclude that $v = 0$. It leads to (3.3) from (3.2).

***Theorem 3.3.*** (Byun & Oh, 2017) *Let* $u \in W^{1,\mathcal{H}}(\Omega)$ *and* $v \in u + W_0^{1,\mathcal{H}}(\Omega_R)$ *is the unique distribution to* (3.1), *then for every* $\gamma > 1$ *there exists a constant* $C = C(\gamma, data) > 0$ *satisfying*

$$\left(\frac{1}{\mathcal{L}^n(\Omega_{R/2})}\int_{\Omega_{R/2}}\left[\mathcal{H}(x,\nabla v)\right]^{\gamma}dx\right)^{\frac{1}{\gamma}} \le \frac{C}{\mathcal{L}^n(\Omega_R)}\int_{\Omega_R}\mathcal{H}(x,\nabla v)dx. \tag{3.4}$$

## 4. Gradient estimates in weighted Lorentz space

### *4.1. Distribution inequalities*

To prove the main results, we will construct some distribution inequalities on level sets as below. From Lemmas 4.1 – 4.3, we always assume that $x_0 \in \Omega, 0 < \mathrm{R} < \mathrm{diam}(\Omega)$, $\omega \in A_\infty$, $u \in W^{1,\mathcal{H}}(\Omega)$ is a distributional solution to $(1.4)$ under assumptions $(1.2),(1.5)$ and $F \in L^{\mathcal{H}}(\Omega)$.

***Lemma 4.1.*** *Given* $0 \le \alpha < n$ *and* $a > 0$, *one can find* $b = b(data) > 0$ *and* $\varepsilon = \varepsilon(data) > 0$ *such that if* $x_1 \in \Omega$ *satisfy* $\mathbf{M}_\alpha(\mathcal{H}(x,F))(x_1) \le \varepsilon^b \lambda$ *for any* $\varepsilon \in (0, \varepsilon_0)$, $\lambda > 0$ *then*

$$d^{\omega}_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\varepsilon^{-a}\lambda\right) \le \varepsilon\omega\left(B_R(0)\right). \tag{4.1}$$

*Proof.* Thanks to Lemma 2.9 with $s = 1$, $f = \mathcal{H}(x,\nabla u) \in L^1(\Omega)$ and Corollary 3.2, we have

$$d_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\varepsilon^{-a}\lambda\right) \le C\left(\frac{1}{\varepsilon^{-a}\lambda}\int_\Omega \mathcal{H}(x,\nabla u)dx\right)^{\frac{n}{n-\alpha}} \le C\left(\frac{1}{\varepsilon^{-a}\lambda}\int_\Omega \mathcal{H}(x,F)dx\right)^{\frac{n}{n-\alpha}}, \tag{4.2}$$

for every $\lambda > 0$. Set $D_0 = \mathrm{diam}(\Omega)$ then $\Omega \subset B_{D_0}(x_1)$, combining with $\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))(x_1) \le \varepsilon^b \lambda$, it follows from (4.2) that

$$\begin{aligned} d_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u)}\left(\varepsilon^{-a}\lambda\right) &\le C\left(\frac{\mathcal{L}^n\left(B_{D_0}(x_1)\right)}{\left(\varepsilon^{-a}\lambda\right)}\cdot\frac{1}{\mathcal{L}^n\left(B_{D_0}(x_1)\right)}\int_{B_{D_0}(x_1)}\mathcal{H}(x,F)dx\right)^{\frac{n}{n-\alpha}} \\ &\le C\left(\frac{D_0^{\,n-\alpha}}{\varepsilon^{-a}\lambda}\mathbf{M}_\alpha\left(\mathcal{H}(x,F)\right)(x_1)\right)^{\frac{n}{n-\alpha}} \\ &\le C\varepsilon^{\frac{(a+b)n}{n-\alpha}}D_0^n \le C\left(\frac{D_0}{R}\right)^n \varepsilon^{\frac{(a+b)n}{n-\alpha}}\mathcal{L}^n\left(B_R(0)\right). \end{aligned} \tag{4.3}$$

By the definition of $\omega \in A_\infty$, (4.3) leads to

$$d^{\omega}_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u)}\left(\varepsilon^{-a}\lambda\right) \le C_0\left(\frac{d_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u)}\left(\varepsilon^{-a}\lambda\right)}{\mathcal{L}^n\left(B_R(0)\right)}\right)^{\delta}\omega\left(B_R(0)\right) \le C_0\left(\frac{D_0}{R}\right)^{n\delta}\varepsilon^{\frac{(a+b)n\delta}{n-\alpha}}\omega\left(B_R(0)\right). \tag{4.4}$$

Let us choose $b > \frac{1}{\delta}\left(1 - \frac{\alpha}{n}\right) - a$ in (4.4) and $\varepsilon_0 = \left[\frac{1}{C_0}\left(\frac{R}{D_0}\right)^{\delta}\right]^{\frac{1}{\frac{(a+b)n\delta}{n-\alpha}-1}}$ to obtain (4.1) for all $\varepsilon \in (0, \varepsilon_0)$. This proof is complete.

***Lemma 4.2.*** *Given* $0 \le \alpha < n,\ a > 0,$ *one can find* $b = b(data)$ *and* $x_2 \in \Omega_R(x_0)$ *satisfy* $\mathbf{M}_\alpha\big(\mathcal{H}(x,F)\big)(x_2) \le \varepsilon^b \lambda$ *for any* $\lambda > 0$. *Then the following inequality*

$$d^\omega_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\Omega_R(x_0), \varepsilon^{-a}\lambda\right) \le d^\omega_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\right)\mathcal{H}(x,\nabla u)}\left(\Omega_R(x_0), \varepsilon^{-a}\lambda\right), \tag{4.5}$$

*holds for any* $\varepsilon \in \left(0, 3^{-\frac{n+1}{a}}\right)$.

*Proof.* Let $\xi \in \Omega_R(x_0)$, based on the definition of fractional maximal operator, we claim that

$$\mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) = \max\left\{\mathbf{M}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi); \mathbf{T}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi)\right\}, \tag{4.6}$$

where

$$\mathrm{M}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) = \sup_{0<r<R} r^\alpha \frac{1}{\mathcal{L}^n\left(B_r(\xi)\right)} \int_{B_r(\xi)} \mathcal{H}(x,\nabla u)dx;$$

$$\mathrm{T}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) = \sup_{r\ge R} r^\alpha \frac{1}{\mathcal{L}^n\left(B_r(\xi)\right)} \int_{B_r(\xi)} \mathcal{H}(x,\nabla u)dx.$$

Of course $B_r(\xi) \subset B_{2R}(x_0)$ with $0 < r < R$, it leads to

$$\mathbf{M}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) \le \mathbf{M}_\alpha\left(\chi_{B_{2R}(x_0)}\mathcal{H}(x,\nabla u)\right)(\xi) \tag{4.7}$$

Furthermore, $B_r(\xi) \subset B_{3r}(x_2)$ with $r \ge R$ and using an assumption $\mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(x_2) \le \lambda,$ it follows that

$$\begin{aligned}\mathbf{T}^R_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) &\le 3^{n-\alpha} \sup_{r\ge R}(3r)^\alpha \frac{1}{\mathcal{L}^n\left(B_{3r}(x_2)\right)} \int_{B_{3r}(x_2)} \mathcal{H}(x,\nabla u)dx \\ &\le 3^n \mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(x_2) \le 3^n\lambda.\end{aligned} \tag{4.8}$$

Combining (4.6), (4.7), and (4.8), we can assert that

$$\left\{\xi \in \Omega_R(x_0) : \left|\mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(\xi)\right| > \varepsilon^{-a}\lambda\right\} \le \left\{\xi \in \Omega_R(x_0) : \left|\mathbf{M}_\alpha\big(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla u)\big)(\xi)\right| > \varepsilon^{-a}\lambda\right\}$$

for all $\varepsilon < 3^{-\frac{n+1}{a}}$. Using the definition of the weighted distribution function, we deduce (4.5).

***Lemma 4.3.*** *Given* $0 \le \alpha < n$ *and* $a > 0,$ *one can find* $b = b(a,\alpha,data) > 0$ *and* $\varepsilon = \varepsilon(a,b,\alpha,data) > 0$ *such that if* $x_1, x_2 \in \Omega_R(x_0)$ *satisfy*

$$\mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(x_1) \le \lambda; \qquad \mathbf{M}_\alpha\big(\mathcal{H}(x,F)\big)(x_2) \le \varepsilon^b\lambda.$$

*then the following inequality*

$$d^\omega_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla u)\right)}\left(\Omega_R(x_0), \varepsilon^{-a}\lambda\right) \le \varepsilon\omega\big(B_R(x_0)\big) \tag{4.9}$$

*holds for all* $\lambda > 0,\ 0 < \varepsilon < \varepsilon_0$.

*Proof.* The main idea of this proof is to apply comparison results and the bounded property of fractional maximal functions to estimate the left-hand side (4.9). Let us fix $x_0 \in \Omega$ and consider $v$ the unique solution to the following problem:

$$\begin{cases} \operatorname{div}\left(\mathcal{A}(x,\nabla v)\right) = 0 & \text{in} \quad \Omega_{4R}(x_0), \\ v = u & \text{on} \quad \partial\Omega_{4R}(x_0). \end{cases}$$

Thanks to estimate (3.2) in Theorem 3.1 and estimate (3.4) in Theorem 3.3, for every $\gamma > 1$, we can state that

$$\left(\frac{1}{\mathcal{L}^n\left(\Omega_{2R}(x_0)\right)}\int_{\Omega_{2R}(x_0)}\left[\mathcal{H}(x,\nabla v)\right]^{\gamma}d\mathrm{x}\right)^{\frac{1}{\gamma}} \le C\,\frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla v)dx, \qquad (4.10)$$

$$\begin{aligned}\frac{1}{\mathcal{L}^n\left(\Omega_{4\mathrm{R}}(x_0)\right)}\int_{\Omega_{4\mathrm{R}}(x_0)}\mathcal{H}(x,\nabla u-\nabla v)dx \le \theta\frac{1}{\mathcal{L}^n\left(\Omega_{4\mathrm{R}}(x_0)\right)}\int_{B_{4\mathrm{R}}(x_0)}\mathcal{H}(x,\nabla u)dx \\ + C\theta^{-\eta}\frac{1}{\mathcal{L}^n\left(\Omega_{4\mathrm{R}}(x_0)\right)}\int_{\Omega_{4\mathrm{R}}(x_0)}\mathcal{H}(x,F)dx.\end{aligned} \qquad (4.11)$$

where $\theta \in (0,1)$ and $\eta = \max\left\{0;\dfrac{2-p}{p-1}\right\}$. Let us denote $K := d_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla u)\right)}\left(\Omega_R(x_0),\varepsilon^{-a}\lambda\right)$. It is easily seen that

$$K \le C\left(d_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla u-\nabla v)\right)}\left(\Omega_R(x_0),\varepsilon^{-a}\lambda\right) + d_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla v)\right)}\left(\Omega_R(x_0),\varepsilon^{-a}\lambda\right)\right). \qquad (4.12)$$

On the right-hand side of (4.12), we apply Lemma 2.9 with $s = 1$ and $s = \gamma > 1$ which will be chosen at the end of the proof, we obtain

$$\begin{aligned} K &\le C\left(\frac{1}{\varepsilon^{-a}\lambda}\int_{\Omega_{2R}(x_0)}\mathcal{H}(x,\nabla u-\nabla v)dx\right)^{\frac{n}{n-\alpha}} + C\left(\frac{1}{\left(\varepsilon^{-a}\lambda\right)^{\gamma}}\int_{\Omega_{2R}(x_0)}[\mathcal{H}(x,\nabla v)]^{\gamma}dx\right)^{\frac{n}{n-\alpha\gamma}} \\ &\le C\left(\frac{(4R)^n}{\varepsilon^{-a}\lambda}\right)^{\frac{n}{n-\alpha}}\left(\frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u-\nabla v)dx\right)^{\frac{n}{n-\alpha}} \\ &\quad + C\left(\frac{(2R)^n}{\left(\varepsilon^{-a}\lambda\right)^{\gamma}}\right)^{\frac{n}{n-\alpha\gamma}}\left(\frac{1}{\mathcal{L}^n\left(\Omega_{2R}(x_0)\right)}\int_{\Omega_{2R}(x_0)}[\mathcal{H}(x,\nabla v)]^{\gamma}dx\right)^{\frac{n}{n-\alpha\gamma}}. \end{aligned} \qquad (4.13)$$

Collecting the estimates (4.10) and (4.11) on the right-hand side of (4.13), one gets that

$$\begin{aligned} K \le C\left(\frac{(4R)^n}{\varepsilon^{-a}\lambda}\right)^{\frac{n}{n-\alpha}} &\left(\frac{\theta}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u)dx + \frac{C\theta^{-\eta}}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,F)dx\right)^{\frac{n}{n-\alpha}} \\ &+ C\left(\frac{(2R)^n}{\left(\varepsilon^{-a}\lambda\right)^{\gamma}}\right)^{\frac{n}{n-\alpha\gamma}}\left(\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla v)dx\right)^{\frac{n}{n-\alpha\gamma}}. \end{aligned} \tag{4.14}$$

We remark that $B_{4R}(x_0) \subset B_{5R}(x_1) \cap B_{5R}(x_2)$ and under assumptions $\mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(x_1) \le \lambda,\ \mathbf{M}_\alpha\big(\mathcal{H}(x,F)\big)(x_2) \le \varepsilon^b\lambda$, it follows

$$\begin{aligned} \frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u)dx &\le \frac{1}{\mathcal{L}^n\left(\Omega_{5R}(x_1)\right)}\int_{\Omega_{5R}(x_1)}\mathcal{H}(x,\nabla u)dx \\ &\le (5R)^{-\alpha}\mathbf{M}_\alpha\big(\mathcal{H}(x,\nabla u)\big)(x_1) \le CR^{-\alpha}\lambda, \\ \frac{1}{\mathcal{L}^n\left(\Omega_{4\mathrm{R}}(x_0)\right)}\int_{\Omega_{4\mathrm{R}}(x_0)}\mathcal{H}(x,F)dx &\le \frac{1}{\mathcal{L}^n\left(\Omega_{5\mathrm{R}}(x_2)\right)}\int_{\Omega_{5R}(x_2)}\mathcal{H}(x,F)dx \\ &\le (5R)^{-\alpha}\mathbf{M}_\alpha\big(\mathcal{H}(x,F)\big)(x_2) \le CR^{-\alpha}\varepsilon^b\lambda. \end{aligned}$$

According to the above estimates, we have

$$\begin{aligned} &\frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla v)dx \\ &\le C\left(\frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u-\nabla v)dx + \frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u)dx\right) \\ &\le C\left(\frac{1}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,\nabla u)dx + \frac{\theta^{-\eta}}{\mathcal{L}^n\left(\Omega_{4R}(x_0)\right)}\int_{\Omega_{4R}(x_0)}\mathcal{H}(x,F)dx\right) \\ &\le CR^{-\alpha}\lambda + CR^{-\alpha}\theta^{-\eta}\varepsilon^b\lambda, \end{aligned}$$

which leads to

$$\xi \in \Omega, r \in (0,R],\ \text{if}\ \ \omega\left(\mathcal{M} \cap B_r(\xi)\right) > \varepsilon\omega(B_r(\xi))\ \text{then}\ \ \Omega_r(\xi) \subset \mathcal{N}.$$

Taking $\theta = \varepsilon^{\frac{b}{1-\eta}}$ and having $1+\theta^{-\eta}\varepsilon^b < 2$, then

$$K \le C\left(\varepsilon^{\left(a+\frac{b}{1+\eta}\right)\frac{n}{n-\alpha}} + \varepsilon^{\frac{an\gamma}{n-\alpha\gamma}}\right)\mathcal{L}^n\left(B_R(x_0)\right). \tag{4.15}$$

Since $\omega \in A_\infty$, it implies from (4.15) that

$$\begin{aligned} d^{\omega}_{\mathbf{M}_\alpha\left(\chi_{B_{2R}}(x_0)\mathcal{H}(x,\nabla u)\right)}\left(\Omega_R(x_0),\varepsilon^{-a}\lambda\right) &\le C_0\left(\frac{K}{\mathcal{L}^n\left(B_R(x_0)\right)}\right)^{\delta}\omega\left(B_R(x_0)\right) \\ &\le C\left(\varepsilon^{\left(a+\frac{b}{1+\eta}\right)\frac{n\delta}{n-\alpha}}+\varepsilon^{\frac{an\gamma\delta}{n-\alpha\gamma}}\right)\omega\left(B_R(x_0)\right). \end{aligned} \tag{4.16}$$

In (4.16), we may choose b and $\gamma$ such that

$$b = an(1+\eta)\left(\frac{\gamma-1}{n-\alpha\gamma}\right) > 0 \text{ and } \frac{n}{\alpha} > \gamma > \max\left\{\frac{n}{na\delta+\alpha};1\right\}$$

to obtain the following estimate

$$d^{\omega}_{\mathbf{M}_\alpha(\chi_{B_{2R}}(x_0))\mathcal{H}(x,\nabla u)}\left(\Omega_R(x_0),\varepsilon^{-a}\lambda\right) \le C\varepsilon^{\frac{an\gamma\delta}{n-\alpha\gamma}}\omega\left(B_R(x_0)\right). \tag{4.17}$$

Let us choose $\varepsilon_0 = \left(\frac{1}{C}\right)^{\frac{1}{\frac{an\gamma\delta}{n-\alpha\gamma}-1}}$ in (4.17) to get (4.9), which completes the proof.

### *4.2. Proofs of main Theorems*

Next, we will introduce a version of covering lemma Calderón-Zygmund (Vitali) ( See Caffarelli and Peral (1998)) for the proof of this lemma.

***Lemma 4.4.*** (Caffarelli & Peral) ***(Covering Lemma)*** *Let $\Omega \subset R^n$ be a bounded domain, $\omega \in A_\infty$ and two measurable sets $\mathcal{M} \subset \mathcal{N} \subset \Omega$. Assume that there exist some constants $\varepsilon \in (0,1)$ and $R \in \left(0, diam\left(\Omega\right)\right)$ satisfying two following hypotheses*

*i)* $\omega\left(\mathcal{M}\right) \le \varepsilon\omega\left(B_R(0)\right)$;

*ii)For any* $\xi \in \Omega, r \in \left(0,R\right],$ if $\omega\left(\mathcal{M} \cap B_r(\xi)\right) > \varepsilon\omega(B_r(\xi))$ then $\Omega_r(\xi) \subset \mathcal{N}$.

*Then there exists a constant* $C = C(n,[\omega]_{\mathbf{A}_\infty}) > 0$ *such that* $\omega\left(\mathcal{M}\right) \le C\varepsilon\omega\left(\mathcal{N}\right)$.

***Proof of Theorem 1.1.*** Firstly, we will prove the inequality

$$\begin{aligned} \omega\Big(\Big\{\xi \in \Omega : \mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) > \varepsilon^{-a}\lambda,\ \mathbf{M}_\alpha\left(\mathcal{H}(x,F)\right)(\xi) \le \varepsilon^{b}\lambda\Big\}\Big) \\ \le C\varepsilon\omega\Big(\Big\{\xi \in \Omega : \mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) > \lambda\Big\}\Big) \end{aligned} \tag{4.18}$$

for any $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$.

Let us introduce two following sets R

$$\mathcal{M}_{\varepsilon,\lambda} = \left\{\xi \in \Omega : \mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) > \varepsilon^{-a}\lambda,\ \mathbf{M}_\alpha\left(\mathcal{H}(x,F)\right)(\xi) \le \varepsilon^{b}\lambda\right\},$$

$$\mathcal{N}_\lambda = \left\{\xi \in \Omega : \mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)(\xi) > \lambda\right\}.$$

We will prove that $\mathcal{M}_{\varepsilon,\lambda}$ and $\mathcal{N}_{\lambda}$ satisfy conditions i, ii of Lemma 4.4, which means $\omega\left(\mathcal{M}_{\varepsilon,\lambda}\right) \le \varepsilon\omega\left(B_R(0)\right)$ for any $\lambda > 0$, $R < diam\left(\Omega\right)$ and for all $\xi \in \Omega$, $r \in \left(0, R\right]$, if $\omega(\mathcal{M}_{\varepsilon,\lambda} \cap B_r(\xi)) > \varepsilon\omega\left(B_r(\xi)\right)$ then $\Omega_r(\xi) \subset \mathcal{N}_{\lambda}$.

It is easy to check that $\mathcal{M}_{\varepsilon,\lambda} = \varnothing$ then it satisfies the two conditions above. If $\mathcal{M}_{\varepsilon,\lambda} \neq \varnothing$ then there exists $x_1 \in \mathcal{M}_{\varepsilon,\lambda}$ such that $\mathbf{M}_{\alpha}\left(\mathcal{H}(x,F)\right)(x_1) \le \varepsilon^b\lambda$. Thanks to Lemma 4.1 with $\varepsilon$ small enough, we conclude that

$$\omega\left(\mathcal{M}_{\varepsilon,\lambda}\right) \le d^{\omega}_{\mathbf{M}_{\alpha}(\mathcal{H}(x,\nabla u))}\left(\varepsilon^{-a}\lambda\right) \le \varepsilon\omega\left(B_R(0)\right),$$

for all $\lambda > 0$ and $R > 0$.

On the other hand, all $\xi \in \Omega$ and $r \in \left(0, R\right]$, let us suppose $\Omega_r(\xi) \not\subset \mathcal{N}_{\lambda}$. We will prove $\omega\left(\mathcal{M}_{\varepsilon,\lambda} \cap B_r(\xi)\right) \le \omega\left(B_r(\xi)\right)$. Since $\Omega_r(\xi) \cap \mathcal{N}^C_{\lambda} \neq \varnothing$, there exists $x_2 \in \Omega_r(\xi)$ satisfying $\mathbf{M}_{\alpha}\left(\mathcal{H}(x,\nabla u)\right)(x_2) \le \lambda$. Moreover, since $\mathcal{M}_{\varepsilon,\lambda} \cap B_r(\xi) \neq \varnothing$, there exists $x_3$ satisfying $\mathbf{M}_{\boldsymbol{\alpha}}\left(\mathcal{H}(x,F)\right)(x_3) \le \varepsilon^b\lambda$. Applying Lemma 4.2 and Lemma 4.3 for $\varepsilon$ small enough one has

$$\begin{aligned}\omega\left(\mathcal{M}_{\varepsilon,\lambda} \cap B_r(\xi)\right) &\le d^{\omega}_{\mathbf{M}_{\alpha}(\mathcal{H}(x,\nabla u))}\left(\Omega_r(\xi), \varepsilon^{-a}\lambda\right) \\ &\le d^{\omega}_{\mathbf{M}_{\alpha}(\chi_{B_{2r}}(\xi)\mathcal{H}(x,\nabla u))}\left(\Omega_r(\xi), \varepsilon^{-a}\lambda\right) \le \varepsilon\omega\left(B_r(\xi)\right),\end{aligned} \tag{4.19}$$

then thanks to Lemma 4.4, we obtain (4.18).

Finally, we observe that

$$\left\{\xi \in \Omega : \mathbf{M}_{\alpha}\left(\mathcal{H}(x,\nabla u)\right)(\xi) > \varepsilon^{-a}\lambda\right\} \subset \mathcal{M}_{\varepsilon,\lambda} \cup \left\{\xi \in \Omega : \mathbf{M}_{\alpha}\left(\mathcal{H}(x,F)\right)(\xi) > \varepsilon^b\lambda\right\},$$

which implies (1.10). The proof is complete.

***Proof of Theorem 1.2.*** Application of Definition 2.7 (Weighted Lorentz space) gives

$$\left\|\mathbf{M}_{\alpha}\left(\mathcal{H}(x,\nabla u)\right)\right\|_{L^{s,t}_{\omega}(\Omega)} = \left[s\int_0^{\infty} \lambda^t \left(d^{\omega}_{\mathbf{M}_{\alpha}(\mathcal{H}(x,\nabla u))}\left(\lambda\right)\right)^{\frac{t}{s}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{t}}. \tag{4.20}$$

For every $0 < s < \infty$ and $0 < t < \infty$ let us fix $0 < a < \dfrac{1}{s}$. Thanks to Theorem 1.1, there exist $b > 0$ and $\varepsilon > 0$ such that (1.8) holds for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$. By changing a variable in the integral of (4.20) combining estimate (1.8), we get that

$$
\begin{aligned}
\left\|\mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)\right\|_{L_\omega^{s,t}(\Omega)}^t &= s\varepsilon^{-at}\int_0^\infty \lambda^t\left(d^\omega_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\varepsilon^{-a}\lambda\right)\right)^{\frac{t}{s}}\frac{d\lambda}{\lambda} \\
&\leq Cs\varepsilon^{-at}\left(\int_0^\infty \varepsilon^{\frac{t}{s}}\lambda^t\left(d^\omega_{\mathbf{M}_\alpha(\mathcal{H}(x,\nabla u))}\left(\lambda\right)\right)^{\frac{t}{s}}\frac{d\lambda}{\lambda}+\int_0^\infty \lambda^t\left(d^\omega_{\mathbf{M}_\alpha(\mathcal{H}(x,F))}\left(\varepsilon^{b}\lambda\right)\right)^{\frac{t}{s}}\frac{d\lambda}{\lambda}\right) \qquad (4.21)\\
&\leq C.\varepsilon^{-at+\frac{t}{s}}\left\|\mathbf{M}_\alpha\left(\mathcal{H}(x,\nabla u)\right)\right\|_{L_\omega^{s,t}(\Omega)}^t + C.\varepsilon^{-at-bt}\left\|\mathbf{M}_\alpha\left(\mathcal{H}(x,F)\right)\right\|_{L_\omega^{s,t}(\Omega)}^t.
\end{aligned}
$$

Since $0<a<\frac{1}{s}$ and $0<t<\infty$, one may choose $\varepsilon\in(0,1)$ satisfying $C\varepsilon^{-at+\frac{t}{s}}\leq\frac{1}{2}$ to obtain (1.11), which completes the proof. It is similar to the case $t=\infty$.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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# MỘT ĐÁNH GIÁ LORENTZ CÓ TRỌNG CHO BÀI TOÁN PHA KÉP

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## TÓM TẮT

*Bài toán pha kép được mô hình từ bài toán cực tiểu một lớp các hàm năng lượng tích phân với điều kiện tăng trưởng không chuẩn. Bài toán này có nhiều ứng dụng trong Vật lí, như trong bài toán đàn hồi phi tuyến, động lực học chất lỏng và các bài toán đồng nhất. Bài báo này đưa ra một đánh giá gradient toàn cục cho nghiệm phân phối của bài toán pha kép trong không gian Lorentz có liên kết với một hàm trọng Muckenhoup. Cụ thể, kết quả này là một dạng đánh giá có trọng so với kết quả chính trong bài báo (Tran & Nguyen, 2021). Phương pháp nghiên cứu của chúng tôi dựa trên việc xây dựng bất đẳng thức hàm phân phối có trọng trên các toán tử cực đại cấp phân số, toán tử này có liên hệ mật thiết với thế vị Riesz.*

***Từ khóa:*** bất đẳng thức hàm phân phối; bài toán pha kép; đánh giá gradient; không gian Lorentz có trọng