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Research Article

CALDERÓN-ZYGMUND COMMUTATORS ON GENERALIZED WEIGHTED LORENTZ SPACES

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ABSTRACT

In this paper, we consider commutators [b, T] of Calderón-Zygmund operators of type θ *(see Definition 1.2 and 1.3 in Section 1) on generalized weighted Lorentz spaces* $\Lambda_u^p(w)$*, where* u *is a function that belongs to the class* A_p *of Muckenhoupt weights on* $\mathbb{R}^n$ *and* w *is a function that belongs to the class* $B_p(u)$ *of Ariño-Muckenhoupt weights on* $(0,\infty)$ *(see Section 1). In this setting, we first establish the pointwise estimate for the sharp maximal operator acting on Calderón-Zygmund commutators of type* θ *(see Lemma 2.2 in Section 2) by using Kolmogorov's inequality, generalized Holder's inequality in the sense of Luxemburg norm (see Definition 2.1) and Young function (see Lemma 2.1), and the well-known John-Nirenberg inequality. In light of this significant estimate, we then indicate that Calderón-Zygmund commutators of type* θ *are bounded on generalized weighted Lorentz spaces* $\Lambda_u^p(w)$ *(see Theorem 2.1) by exploiting the ideas and techniques concerning maximal operators from the study by Carro et al. (2021). Our aforementioned main results extend the ones of Carro et al. (2021).*

Keywords: Ariño and Muckenhoupt weights; Calderón-Zygmund commutators; generalized weighted Lorentz spaces; maximal operators

1. Introduction

Let $1<p<\infty$, $b\in BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular operator. In 1976, the boundedness of the commutator $[b,T]f=bT(f)-T(bf)$ on Lebesgue spaces L^p was first studied by Coifman et al. (1976), that is,

$$\|[b,T]f\|_{L^p}=\|bT(f)-T(bf)\|_{L^p}\leqslant C\|b\|_{BMO}\|f\|_{L^p},$$

where C is a positive constant independent of b and f.

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Since then, many papers concerning the boundedness of Calderón-Zygmund operators and commutators, as well as their applications in partial differential equations, operator theory, and complex analysis, have been published (see Coifman et al., 1976 and Taylor, 2000). In 1994, the boundedness of commutators on $L^p(u)$ was established (Alvarez et al., 1994) when u belongs to the class A_p, which means that

$$\left(\frac{1}{|B|}\int_B u(x)dx\right)\left(\frac{1}{|B|}\int_B \left(u(x)\right)^{-1/p-1}\right)^{p-1} < \infty,$$

for every ball $B \subset \mathbb{R}^n$ (Muckenhoupt, 1972).

These results were then extended to generalized weighted Lorentz spaces $\Lambda_u^p(w)$ by Carro et al. (2021). More specifically, the authors showed that $[b,T]$ is bounded on $\Lambda_u^p(w)$ if $u \in A_\infty$, $w \in B_p(u) \cap B_\infty^*$ and $b \in BMO$, where A_∞, $B_p(u)$, B_∞^* were first defined by Muckenhoupt (1972), Grafakos (2009), and Alvarez and Pérez (1994) respectively.

For convenience, we recall here the definition of generalized weighted Lorentz spaces $\Lambda_u^p(w)$.

Definition 1.1. *(Carro et al., 2007) Let u be a weight on $\mathbb{R}^n$ and w be a weight on $(0,\infty)$. For every $p > 1$, the generalized weighted Lorentz space $\Lambda_u^p(w)$ is the set of all functions f satisfying*

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty f_u^*(t)^p w(t)dt\right)^{1/p} < \infty,$$

where f_u^* is the decreasing rearrangement of f, which is defined by

$$f_u^*(t) = \inf\{s : \lambda_f^u(s) \le t\}, t \ge 0,$$

and

$$\lambda_f^u(s) = u(\{x \in \mathbb{R}^n : |f(x)| > s\}), s \ge 0$$

is the distribution function of f with respect to the measure $u(x)dx$.

Inspired by the above works, we aim to extend the main result by Carro et al. (2021) to the setting of Calderón-Zygmund commutators of type θ on $\Lambda_u^p(w)$.

Definition 1.2. *(Grafakos, 1985) Let θ be a nonnegative, nondecreasing function on $(0,\infty)$ with $\int_0^1 \theta(t)t^{-1}dt < \infty$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ is said to be a standard kernel of type θ if it satisfies the following conditions.*

(i) (Size condition)

$$|K(x,y)| \le \frac{C}{|x-y|^n}. \tag{1.1}$$

(ii) (Regularity condition)

$$|K(x,y)-K(x_0,y)|+|K(y,x)-K(y,x_0)|\le C|x_0-y|^{-n}\,\theta\left(\frac{|x_0-x|}{|y-x_0|}\right), \tag{1.2}$$

for every x, x_0, y *with* $2|x-x_0|<|y-x_0|$.

Definition 1.3. *(Grafakos, 1985) Let* θ *be a function as in Definition 1.2. A linear operator* T *from* $\mathcal{S}(\mathbb{R}^n)$ *to* $\mathcal{S}'(\mathbb{R}^n)$ *is said to be a Calderón-Zygmund operator of type* θ *if it satisfies the following conditions.*

(i) T *is bounded on* $L^2(\mathbb{R}^n)$*, which means*

$$\|Tf\|_{L^2}\le C\|f\|_{L^2}, \quad \text{for every } f\in C_0^\infty(\mathbb{R}^n). \tag{1.3}$$

(ii) There exists a standard kernel K *of type* θ *such that for every function* $f\in C_0^\infty(\mathbb{R}^n)$ *and* $x\notin supp(f)$

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy. \tag{1.4}$$

The structure of our paper is as follows. We establish pointwise estimates for sharp maximal operators and key lemmas in Section 2. Then, we prove the boundedness of Calderón-Zygmund of commutators of type θ on $\Lambda_u^p(w)$.

As usual, we use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line and C_p to denote a positive constant that is dependent on subscript p. If $f\le Cg$, we write $f\lesssim g$ or $g\gtrsim f$; and if $f\lesssim g\lesssim f$, we write $f\sim g$.

2. Main results

2.1. Pointwise estimates for sharp maximal operators

For $\beta>0$, let M_β be the modified Hardy–Littlewood maximal function

$$M_\beta f(x)=M(|f|^\beta)^{1/\beta}(x)=\left(\sup_{r>0}\frac{1}{|B|}\int_B|f(y)|^\beta\,dy\right)^{1/\beta},$$

and let $M_\beta^\sharp$ be the modified sharp maximal function

$$M_\beta^\sharp f(x)=\sup_{r>0}\inf_{c\in\mathbb{R}}\left(\frac{1}{|B|}\int_B\big||f(y)|^\beta-|c|^\beta\big|dy\right)^{1/\beta},$$

where $B=B(x,r)$ is a ball in $\mathbb{R}^n$.

Definition 2.1. *(Liu et al., 2002) A function* $\phi:[0,\infty)\to[0,\infty)$ *is a Young function if it is continuous, convex, increasing, and satisfying* $\phi(0)=0$*,* $\phi(t)\to\infty$ *as* $t\to\infty$.

We define the ϕ-average of a function f over a ball B by means of the following Luxemburg norm

$$\| f \|_{\phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \phi\left(\frac{|f(y)|}{\lambda}\right)dy \leqslant 1\right\}.$$

The following lemma is the generalized Holder inequality.

Lemma 2.1. *(Liu et al., 2002) Suppose that ϕ is a Young function, f and g are measurable functions such that $\| f \|_{\phi,B} < \infty$, $\| g \|_{\bar{\phi},B} < \infty$. Then,*

$$\frac{1}{|B|}\int_B |f(y)g(y)|\,dy \leqslant \| f \|_{\phi,B} \| g \|_{\bar{\phi},B},$$

where $\bar{\phi}$ is the complementary Young function associated to ϕ, which is defined as

$$\bar{\phi}(x) = \sup_{y\geq 0}\{xy - \phi(y)\}.$$

In particular, when $\phi(t) = t\left(1+\log^+ t\right)$, its complementary Young function is $\overline{\phi(t)} \approx \exp(t)$. In this situation, we denote

$$\| f \|_{\phi,B} = \| f \|_{L\log L,B}, \quad \| g \|_{\bar{\phi},B} = \| g \|_{\exp L,B}.$$

The maximal function associated with $\phi(t) = t(1+\log^+(t))$ is defined by

$$M_{L\log L} f(x) = \sup_{x\in B} \| f \|_{L\log L,B}.$$

Lemma 2.2. *Let T be a Calderón-Zygmund operator of type θ, $b \in BMO(\mathbb{R}^n)$ and $0 < \beta < \epsilon < 1$. Then, there is a constant $C > 0$ such that*

$$M_\beta^\sharp\left([b,T](f)\right)(x_0) \leqslant C\|b\|_{BMO}\left(M_\epsilon(T(f))(x_0) + M^2(f)(x_0)\right),$$

for all bounded functions f with compact support and $x_0 \in \mathbb{R}^n$, where $M^2(f) = M(M(f))$.

The following proof is motivated by the proof of Lemma 3 in (Liu et al., 2002).

Proof. First, we prove for each $0 < \beta < 1$, each ball $B = B(x_0, r)$, and for some constant $c = c_\beta$, there exists $C = C_\beta > 0$ such that

$$\left(\frac{1}{|B|}\int_B \left||[b,T]f(y)|^\beta - |c|^\beta\right|dy\right)^{1/\beta} \leqslant C\|b\|_{BMO}\left(M_\epsilon(Tf)(x_0) + M^2 f(x_0)\right).$$

Let $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 2B}$. Then, we have

$$[b,T]f = (b - b_{2B})Tf - T((b-b_{2B})f_1) - T((b-b_{2B})f_2).$$

We pick $c = -(T((b-b_{2B})f_2))_B$, it follows from the following inequalities

$$\left||a|^\beta - |b|^\beta\right| \leq |a-b|^\beta,$$

$$|a+b|^\beta \lesssim |a|^\beta + |b|^\beta,$$

that

$$\left(\frac{1}{|B|}\int_B \left| |[b,T]f(y)|^\beta - |-(T((b-b_{2B})f_2))_B|^\beta \right| dy\right)^{1/\beta} \le \left(\frac{1}{|B|}\int_B |[b,T]f(y)+(T((b-b_{2B})f_2))_B|^\beta \, dy\right)^{1/\beta}$$

$$=\left(\frac{1}{|B|}\int_B |(b(y)-b_{2B})Tf(y)-T((b-b_{2B})f_1)(y)-T((b-b_{2B})f_2)(y)+(T((b-b_{2B})f_2))_B|^\beta\right)^{1/\beta}$$

$$\lesssim \left(\frac{1}{|B|}\int_B |b(y)-b_{2B}|^\beta |Tf(y)|^\beta \, dy\right)^{1/\beta} + \left(\frac{1}{|B|}\int_B |T((b-b_{2B})f_1)(y)|^\beta \, dy\right)^{1/\beta}$$

$$+\left(\frac{1}{|B|}\int_B |T((b-b_{2B})f_2)(y)-(T((b-b_{2B})f_2))_B|^\beta \, dy\right)^{1/\beta}.$$

Set $I_1=\left(\frac{1}{|B|}\int_B |b(y)-b_{2B}|^\beta |Tf(y)|^\beta \, dy\right)^{1/\beta}$, $I_2=\left(\frac{1}{|B|}\int_B |T((b-b_{2B})f_1)(y)|^\beta \, dy\right)^{1/\beta}$ and

$$I_3=\left(\frac{1}{|B|}\int_B |T((b-b_{2B})f_2)(y)-(T((b-b_{2B})f_2))_B|^\beta \, dy\right)^{1/\beta}.$$

For I_1, since $1<r<\epsilon/\beta$, applying Holder's inequality with exponents r and r', we get

$$I_1 \lesssim \left(\frac{1}{|B|}\int_B |Tf(y)|^{\beta r} \, dy\right)^{1/\beta r}\left(\frac{1}{|B|}\int_B |b(y)-b_{2B}|^{\beta r'} \, dy\right)^{1/\beta r'}$$

$$\lesssim \left(\frac{1}{|B|}\int_B |Tf(y)|^{\beta r} \, dy\right)^{1/\beta r}\left(\frac{1}{|2B|}\int_{2B} |b(y)-b_{2B}|^{\beta r'} \, dy\right)^{1/\beta r'}$$

$$\lesssim \|b\|_{BMO} M_{\beta r}(Tf)(x_0)$$

$$\lesssim \|b\|_{BMO} M_\epsilon(Tf)(x_0).$$

For I_2, since T is an operator of weak type (1,1) and $0<\beta<1$, so according to Kolmogorov's inequality and generalized Holder's inequality, we get

$$I_2 \lesssim \frac{1}{|B|}\int_{\mathbb{R}^n} |(b-b_{2B})f_1(y)|\,dy$$

$$\lesssim \frac{1}{|B|}\int_{2B} |(b-b_{2B})f(y)|\,dy$$

$$\lesssim \|b-b_{2B}\|_{\exp L,2B}\|f\|_{L\log L,2B}.$$

Next, we claim the following: there is a positive constant C such that for all balls B,

$$\|b-b_{2B}\|_{\exp L,2B} \lesssim \|b\|_{BMO}. \tag{2.1}$$

Indeed, this is equivalent to

$$\frac{1}{|2B|}\int_{2B}\exp\left(\frac{|b(y)-b_{2B}|}{C\|b\|_{BMO}}\right)dy \le C_0,$$

which is the fundamental estimate of John and Nirenberg (1961).

On the other hand, it follows from the following definition of $M_{L\log L}f(x_0)$, that

$$\|f\|_{L\log L,2B} \leqslant M_{L\log L}f(x_0).$$

Hence,

$$I_2 \lesssim \|b\|_{BMO} M_{L\log L}f(x_0).$$

To estimate I_3 we use Holder's inequality with $0<\beta<1$. Then

$$I_3 = \frac{1}{|B|^{1/\beta}}\frac{1}{|B|^{1-1/\beta}}\left(\int_B |T((b-b_{2B})f_2)-(T((b-b_{2B})f_2)_B|^{\beta}\,dx\right)^{1/\beta}\left(\int_B dx\right)^{1-1/\beta}$$

$$\leqslant \frac{1}{|B|}\int_B |T((b-b_{2B})f_2)-(T((b-b_{2B})f_2))_B|\,dx$$

$$\leqslant \frac{1}{|B|}\int_B\left|\int_{\mathbb{R}^n}K(x,y)(b(y)-b_{2B})f_2(y)dy-\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(z,y)(b(y)-b_{2B})f_2(y)dydz\right|dx$$

$$\leqslant \frac{1}{|B|}\int_B\left|\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(x,y)(b(y)-b_{2B})f_2(y)dydz-\frac{1}{|B|}\int_B\int_{\mathbb{R}^n}K(z,y)(b(y)-b_{2B})f_2(y)dydz\right|dx$$

$$\leqslant \frac{1}{|B|}\frac{1}{|B|}\int_B\left|\int_B\int_{\mathbb{R}^n}(b(y)-b_{2B})f_2(y)\big(K(x,y)-K(z,y)\big)dydz\right|dx$$

$$\leqslant \frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n}|(b(y)-b_{2B})f_2(y)|\,|K(x,y)-K(z,y)|\,dydzdx.$$

$$=\frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n\setminus 2B}|(b-b_{2B})f(y)|\,|K(x,y)-K(z,y)|\,dydzdx.$$

Next, we have the following estimates

$$|b_{2^{j+1}B}-b_{2B}| \leqslant \sum_{i=1}^{j}|b_{2^{j+1}B}-b_{2^jB}| \leqslant \sum_{i=1}^{j}\left|\frac{1}{|2^jB|}\int_{2^jB}b(y)dy-b_{2^{j+1}B}\right|$$

$$\leqslant \sum_{i=1}^{j}2^n\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|\,dy$$

$$\lesssim \sum_{i=1}^{j}\|b\|_{BMO}$$

$$\lesssim j\|b\|_{BMO}.$$

Take $z,\ x\in B$ and $y\notin 2B$. Then, we have $2|x-x_0|<|y-x_0|$ and $2|z-x_0|<|y-x_0|$. It follows from the regularity condition of the definition of the standard kernel of type θ that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B}|(b(y)-b_{2B})f(y)|\left|K(x,y)-K(x_0,y)\right|dy\\
&\lesssim\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{\theta(|x-x_0|/|y-x_0|)}{|x_0-y|^n}|(b(y)-b_{2B})f(y)|\,dy\\
&\lesssim\sum_{j=1}^{\infty}\theta(2^{-j})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(y)-b_{2B})f(y)|\,dy\\
&\lesssim\sum_{j=1}^{\infty}\theta(2^{-j})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(y)-b_{2^{j+1}B})||f(y)|\,dy\\
&+\sum_{j=1}^{\infty}\theta(2^{-j})|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy
\end{aligned}$$

Using the inequality (7), $Mf\left(x_0\right)\le M_{L\log L}f\left(x_0\right)$ and the generalized Holder's inequality, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\theta(2^{-j})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(y)-b_{2^{j+1}B})||f(y)|\,dy+\sum_{j=1}^{\infty}\theta(2^{-j})|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy\\
&\lesssim\sum_{j=1}^{\infty}\theta(2^{-j})\left\|b-b_{2^{j+1}B}\right\|_{\exp L,2^{j+1}B}\|f\|_{L\log L,2^{j+1}B}+\sum_{j=1}^{\infty}j\theta(2^{-j})\|b\|_{BMO}Mf(x_0)\\
&\lesssim\sum_{j=1}^{\infty}j\theta(2^{-j})\|b\|_{BMO}M_{L\log L}f(x_0)\\
&\lesssim\int_0^1\theta(t)t^{-1}dt\,\|b\|_{BMO}M_{L\log L}f(x_0)\\
&\lesssim\|b\|_{BMO}M_{L\log L}f(x_0).
\end{aligned}$$

By an argument analogous to $|K(z,y)-K(x_0,y)|$, we obtain

$$\int_{\mathbb{R}^n\setminus 2B}|(b(y)-b_{2B})f(y)|\left|K(z,y)-K(x_0,y)\right|dy\lesssim\|b\|_{BMO}M_{L\log L}f(x_0).$$

Hence,

$$I_3 \leqslant \frac{1}{|B|}\frac{1}{|B|}\int_B \int_B \int_{\mathbb{R}^n\setminus 2B} |(b-b_{2B})f(y)|\left|K(x,y)-K(z,y)\right| dydzdx.$$

$$\lesssim \frac{1}{|B|}\frac{1}{|B|}\int_B \int_B \int_{\mathbb{R}^n\setminus 2B} |(b(y)-b_{2B})f(y)|\left(\left|K(x,y)-K(x_0,y)\right|+\left|K(z,y)-K(x_0,y)\right|\right) dydzdx.$$

$$\lesssim \frac{1}{|B|}\frac{1}{|B|}\int_B \int_B \|b\|_{BMO} M_{L\log L} f(x_0) dzdx.$$

$$\lesssim \|b\|_{BMO} M_{L\log L} f(x_0).$$

Finally, since $M^2 f \approx M_{L\log L}$ (Grafakos, 2009, Lemma 7.5.4), we have

$$\left(\frac{1}{|B|}\int_B |[b,T]f(y)-c|^\beta\, dy\right)^{1/\beta} \leqslant I_1+I_2+I_3$$

$$\lesssim \|b\|_{BMO}\left(M_\epsilon(Tf)(x_0)+M_{L\log L} f(x_0)\right)$$

$$\lesssim \|b\|_{BMO}\left(M_\epsilon(Tf)(x_0)+M^2(f)(x_0)\right),$$

which completes the proof of Lemma 2.3.

2.2. Main Result

In the sequel, we need the following lemmas.

First, we recall here the boundedness of the Hardy-Littlewood maximal operator M on $\Lambda_u^p(w)$.

Lemma 2.3. *(Carro et al., 2007, Theorem 3.3.5)* Let $1 < p < \infty$ *and let u, w be weights in* $\mathbb{R}^n$ *and* $\mathbb{R}^+$*, respectively. M be the Hardy-Littlewood maximal operator. Then,* M is bounded on $\Lambda_u^p(w)$ if and only if $w \in B_p(u)$.

Lemma 2.4. *(Carro et al., 2007, Proposition 2.2.12 and Lemma 3.3.1) If* $w \in B_p(u)$ *then W,* where $W(t)=\int_0^t w(s)ds$, $t>0$, *holds the following conditions:*

(i) $W(2t) \lesssim W(t), \forall t>0$,

(ii) $W(s+t) \lesssim W(s)+W(t), \forall s,t>0$.

The condition (i) is also known as the doubling condition.

Lemma 2.5. *(Carro et al., 2021, Lemma 2.6) Assume that* $1 < p < \infty$, $u \in A_\infty$, $w \in B_\infty^*$, *and W satisfies the doubling condition. Then*

$$\| Mf \|_{\Lambda_u^p(w)} \lesssim \| M^\sharp f \|_{\Lambda_u^p(w)},$$

provided that $\| Mf \|_{\Lambda_u^p(w)} < \infty$.

The next two lemmas are known as Fatou's lemma in the weighted Lorentz spaces and the John-Nirenberg inequality, respectively.

Lemma 2.6. *(Carro et al., 2007, Proposition 2.2.8) Let* $0 < p \le \infty$ *and* f_k, $k \ge 1$, *be Lebesgue measurable functions. Then*

$$\left\| \liminf_k |f_k| \right\|_{\Lambda_u^p(w)} \le \liminf_k \| f_k \|_{\Lambda_u^p(w)}.$$

We also need the boundedness of Calderón-Zygmund operators on $\Lambda_u^p(w)$.

Lemma 2.7. *Assume* $f \in BMO$. *Then, for every ball* $B \subset \mathbb{R}^n$, *there exists constants* C_1 *and* C_2, *dependent only on the dimension* n, *such that*

$$\left| \left\{ x \in B : |f(x) - f_B| > t \right\} \right| \le C_1 |B| \exp\left(-\frac{C_2 t}{\|f\|_{BMO}} \right), \ t > 0.$$

Theorem 2.1. *(Thai et al., 2022, Theorem 2.1) Let* T *be a Calderón-Zygmund operator of type* $\theta, 1 < p < \infty$, $u \in A_\infty$ *and* $w \in B_\infty^* \cap B_p(u)$. *Then* T *is bounded on* $\Lambda_u^p(w)$.

Now we are ready to prove the following theorem.

Theorem 2.2. Let $1 < p < \infty$, $b \in BMO$ and T be a Calderón-Zygmund operator of type θ. If $u \in A_\infty$ and $w \in B_\infty^* \cap B_p(u)$ then

$$\left\| [b, T](f) \right\|_{\Lambda_u^p(w)} \le C \|b\|_{BMO} \|f\|_{\Lambda_u^p(w)}.$$

Proof. It suffices to prove Theorem 2.2 when f is a bounded function with compact support. Our proof includes two steps.

Step 1. The function b is bounded on $\mathbb{R}^n$.

Let $0 < \beta < \varepsilon < 1$. Since T is bounded on $\Lambda_u^p(w)$, Tf is in $\Lambda_u^p(w)$. In addition, as b is a bounded function on $\mathbb{R}^n$, bTf is in $\Lambda_u^p(w)$. Therefore, by Lemma 2.1, we have

$$\left\| [b, T](f) \right\|_{\Lambda_u^p(w)} = \left\| bT(f) - T(bf) \right\|_{\Lambda_u^p(w)} \lesssim \left\| bT(f) \right\|_{\Lambda_u^p(w)} + \left\| T(bf) \right\|_{\Lambda_u^p(w)} < \infty.$$

Since $w \in B_p(u)$, M_β is bounded on $\Lambda_u^p(w)$. Hence, combining the above estimate, we get $\left\| M_\beta [b, T] f \right\|_{\Lambda_u^p(w)}$ is finite. Then applying Lemma 2.5 gives

$$\left\| M_\beta [b, T] f \right\|_{\Lambda_u^p(w)} \lesssim \left\| M_\beta^\# [b, T] f \right\|_{\Lambda_u^p(w)}.$$

On the other hand, by Lemma 2.2 and the boundedness of M and M_ε on $\Lambda_u^p(w)$, we obtain

$$\begin{aligned} \left\| M_\beta^\# [b, T] f \right\|_{\Lambda_u^p(w)} &\lesssim \|b\|_{BMO} \left(\left\| M_\varepsilon (Tf) \right\|_{\Lambda_u^p(w)} + \left\| M^2 (f) \right\|_{\Lambda_u^p(w)} \right) \\ &\lesssim \|b\|_{BMO} \left(\|Tf\|_{\Lambda_u^p(w)} + \|f\|_{\Lambda_u^p(w)} \right). \end{aligned}$$

Again, since T is bounded on $\Lambda_u^p(w)$, which means $\|Tf\|_{\Lambda_u^p(w)} \lesssim \|f\|_{\Lambda_u^p(w)}$, so

$$\left\| M_\beta^\# [b, T] f \right\|_{\Lambda_u^p(w)} \lesssim \|b\|_{BMO} \|f\|_{\Lambda_u^p(w)}.$$

Notice that $|f(x)| \le |M_\beta(x)|$ for almost every $x \in \mathbb{R}^n$, we get

$$\left\| [b, T](f) \right\|_{\Lambda_u^p(w)} \le \left\| M_\beta [b, T] f \right\|_{\Lambda_u^p(w)} \lesssim \left\| M_\beta^\# [b, T] f \right\|_{\Lambda_u^p(w)} \lesssim \|b\|_{BMO} \|f\|_{\Lambda_u^p(w)}.$$

Step 2. b is a BMO function.

Let us consider the following increasing sequence of functions

$$b_k(x) = \begin{cases} k, & b(x) > k, \\ b(x), & -k \le b(x) \le k, \\ -k, & b(x) < -k. \end{cases}$$

It is clear that b_k are bounded functions and that $\|b_k\|_{BMO} \le \frac{9}{8}\|b\|_{BMO}$.

On the other hand, since f has compact support, we choose a ball B such that supp $f \subset B$. Then, by the John-Nirenberg inequality for $p = 2$, for every $t > 0$, there exists $C > 0$ such that

$$\int_B |b(x) - b_B|^2 dx = 2\int_0^\infty t \left| \{ x \in B : |b(x) - b_B| > t \} \right| dt$$

$$\lesssim 2\int_0^\infty t |B| \exp\left(-\frac{Ct}{\|b\|_{BMO}} \right) dt \lesssim 2\int_0^\infty u \exp(-u)\, du < \infty.$$

This leads to $b - b_B \in L^2(B)$, and hence $b - b_B \in L^2$(supp f). Therefore, $b \in L^2$(supp f).

Now by the triangle inequality, we obtain

$$|b_k(x) - b(x)|^2 \le \left(|b_k(x)| + |b(x)| \right)^2 \le \left(2|b(x)| \right)^2.$$

Then, observe that b_k converges pointwise to b, it follows from the Lebesgue dominated convergence theorem that b_k converges to b in L^2(supp f). Consequently, $b_k f$ converges to bf in $L^2(\mathbb{R}^n)$. According to the boundedness of T on $L^2(\mathbb{R}^n)$, we have

$$\|T(b_k f) - T(bf)\|_2 \lesssim \|b_k f - bf\|_2,$$

which implies that $T(b_k f)$ converges to $T(bf)$ in $L^2(\mathbb{R}^n)$.

Thus, there exists a subsequence b_{k_j} of b_k such that $T(b_{k_j} f)$ converges to $T(bf)$ a.e. on $\mathbb{R}^n$. This leads to

$$\left[b_{k_j}, T \right](f)(x) = b_{k_j}(Tf)(x) - T(b_{k_j} f)(x) \to b(Tf)(x) - T(bf)(x) = [b, T](f)(x) \quad \text{as } j \to \infty$$

for almost every $x \in \mathbb{R}^n$.

Finally, in view of Fatou's lemma for weighted Lorentz spaces, we get

$$\begin{aligned}\|[b,T]f\|_{\Lambda_u^p(w)} &= \left\|\liminf\left[b_{k_j},T\right]f\right\|_{\Lambda_u^p(w)}\\ &\le \liminf\left\|\left[b_{k_j},T\right]f\right\|_{\Lambda_u^p(w)}\\ &\lesssim \liminf\left\|b_{k_j}\right\|_{BMO}\|f\|_{\Lambda_u^p(w)}\\ &\lesssim \|b\|_{BMO}\|f\|_{\Lambda_u^p(w)},\end{aligned}$$

where we use the result of Step 1 for the second inequality since b_{k_j} are bounded functions.

3. Conclusion

In summary, using the ideas and techniques of Carro et al. (2021), we obtain the following pointwise estimate for the sharp maximal operator (Lemma 2.2) and the boundedness of Calderón-Zygmund commutators of type θ on the generalized weighted Lorentz spaces $\Lambda_u^p(w)$ (Theorem 2.1):

Main Result 1. *Let T be a Calderón-Zygmund operator of type θ, $b \in BMO(\mathbb{R}^n)$ and $0<\beta<\epsilon<1$. Then, there is a constant $C>0$ such that*

$$M_\beta^\sharp\left([b,T](f)\right)(x_0) \leqslant C\|b\|_{BMO}\left(M_\epsilon(T(f))(x_0)+M^2(f)(x_0)\right),$$

for all bounded functions f with compact support and $x_0\in\mathbb{R}^n$, where $M^2(f)=M\left(M(f)\right)$.

Main Result 2. Let $1<p<\infty$, $b\in BMO$ and T be a Calderón-Zygmund operator of type θ. If $u\in A_\infty$ and $w\in B_\infty^*\cap B_p(u)$ then

$$\|[b,T](f)\|_{\Lambda_u^p(w)} \le C\|b\|_{BMO}\|f\|_{\Lambda_u^p(w)}.$$

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HOÁN TỬ CALDERÓN-ZYGMUND TRÊN CÁC KHÔNG GIAN LORENTZ TỔNG QUÁT

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TÓM TẮT

Trong bài báo này, chúng tôi xét hoán tử của toán tử Calderón-Zygmund loại θ *(xem Định nghĩa 1.2 và 1.3 trong Phần 1) trên các không gian Lorentz tổng quát* $\Lambda_u^p(w)$*, trong đó* u *là một hàm thuộc lớp hàm trọng Muckenhoupt* A_p *trên* $\mathbb{R}^n$ *và* w *là một hàm thuộc lớp hàm trọng Ariño-Muckenhoupt* $B_p(u)$ *trên* $(0,\infty)$ *(xem Phần 1). Trên cấu hình này, trước tiên chúng tôi thiết lập đánh giá từng điểm cho toán tử cực đại nhọn tác động lên hoán tử Calderón-Zygmund loại* θ *(xem Bổ đề 2.2, Phần 2) bằng cách sử dụng bất đẳng thức Kolmogorov, bất đẳng thức Holder tổng quát theo chuẩn* Luxemburg *(xem Định nghĩa 2.1) của các hàm Young (xem Bổ đề 2.1) và bất đẳng thức John-Nirenberg nổi tiếng. Nhờ có đánh giá quan trọng này, chúng tôi sau đó chỉ ra rằng hoán tử Calderón-Zygmund loại* θ *bị chặn trên các không gian Lorentz tổng quát (xem Định lí 2.1) bằng cách khai thác các ý tưởng cũng như kĩ thuật của (Carro et al., 2021). Các kết quả chính nêu trên của chúng tôi mở rộng các kết quả tương ứng trong bài báo của Carro et al., 2021.*

Từ khoá: hàm trọng Ariño và Muckenhoupt; hoán tử Calderón-Zygmund loại θ; không gian Lorentz có trọng tổng quát; toán tử cực đại