TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 19, Số 10 (2022): 1756-1767
Website: https://journal.hcmue.edu.vn
ISSN: 2734-9918

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 19, No. 10 (2022): 1756-1767
https://doi.org/10.54607/hcmue.js.19.10.3566(2022)

Research Article

EXISTENCE AND UNIQUENESS RESULTS FOR PARAMETRIC MIXED VARIATIONAL-HEMIVARIATIONAL PROBLEMS

Vo Minh Tam
Dong Thap University, Vietnam
Corresponding author: Vo Minh Tam– Email: vmtam@dthu.edu.vn
Received: August 21, 2022; Revised: October 17, 2022; Accepted: October 31, 2022

ABSTRACT

In this paper, we introduce a general class of parametric mixed variational-hemivariational problems involving the Clarke's generalized derivatives and equilibrium functions (for brevity, PMVHP). Then, based on the technique involving the Tarafdar's fixed point theorem and some arguments in the nonsmooth analysis, the existence of solutions for the problem PMVHP is studied. Furthermore, we establish the uniqueness of the solution to the problem PMVHP under some strong monotonicity assumptions. Our main results in this paper extend the corresponding results in Matei (2019, 2022).

Keywords: Clarke's generalized derivative; existence and uniqueness; fixed points for set-valued mappings; parametric mixed variational-hemivariational problem

1. Introduction and Preliminaries

Let $\left(X,(\cdot,\cdot)_X,\|\cdot\|_X\right)$ and $\left(\Omega,(\cdot,\cdot)_\Omega,\|\cdot\|_\Omega\right)$ be two Hilbert spaces, Γ be a metric space, $Z\subset X$ and $\Lambda\subset\Omega$ be nonempty subsets, $A:X\times X\to\mathbb{R}$ and $B:X\times\Omega\to\mathbb{R}$ be bilinear functions, $\eta:\Gamma\times X\to X$ be continuous operator, $h:Z\times Z\to\mathbb{R}$ and $k:\Lambda\times\Lambda\to\mathbb{R}$ satisfying $h(z,z)=0$ and $k(\lambda,\lambda)=0,\ \forall z\in Z,\forall\lambda\in\Lambda,\ P:X\to X_P$ and $Q:\Omega\to\Omega_Q$ be linear operators with $\left(X_P,(\cdot,\cdot)_{X_P},\|\cdot\|_{X_P}\right)$ and $\left(\Omega_Q,(\cdot,\cdot)_{\Omega_Q},\|\cdot\|_{\Omega_Q}\right)$ being Hilbert spaces, $\Phi:X_P\to\mathbb{R}$ and $\Upsilon:\Omega_Q\to\mathbb{R}$ be Lipschitz continuous functions. The goal of this paper is to investigate the following parametric mixed variational-hemivariational problem involving the Clarke's generalized derivatives and equilibrium functions:

Cite this article as: Vo Minh Tam (2022). Existence and uniqueness results for parametric mixed variational-hemivariational problems. *Ho Chi Minh City University of Education Journal of Science, 19*(10), 1756-1767.

Problem 1.1. Given $\gamma \in \Gamma$ and $f \in X$, find $(z,\lambda) \in Z \times \Lambda$ such that

$$\begin{cases} A\big(\eta(\gamma,z), v-z\big) + B(v-z,\lambda) + h(z,v) + \Phi^0(Pz; Pv-Pz) \ge (f, v-z)_X \\ B(z,\mu-\lambda) - k(\lambda,\mu) - \Upsilon^0(Q\lambda; Q\mu - Q\lambda) \le 0, \end{cases}$$

For all $(v,\mu) \in Z \times \Lambda$, where $\Phi^0(p;v)$ (resp., $\Upsilon^0(q;w)$) stands for Clarke's generalized derivative of Φ at $p \in X_P$ (resp., $q \in \Omega_Q$) with respect to the direction $v \in X_P$ (resp., $w \in \Omega_Q$).

It is well known that the theory of variational-hemivariational inequalities is a generalization of hemivariational inequalities and variational inequalities involving both the convex and the nonconvex potentials and based on the Clarke's generalized gradient for locally Lipschitz functions. This theory was developed in the early 1980s in mechanics and has found various complex problems in mechanics and engineering, especially in optimization and nonsmooth analysis(see Panagiotopoulos,1985, 1993). Many authors have extensively studied this theory in different directions, such as the existing results, the solution method, the stability, the well-posedness, and the error bound (see Han et al., 2014; Sofonea & Danan, 2018; Nguyen et al., 2020a, 2020b, 2021; Vo, 2022). Very recently, various mixed variational-hemivariational problems have been studied to their existence and their applications (Matei, 2019; Migórski et al., 2019; and Bai et al., 2020). Matei (2022) recently developed the theory of mixed variational-hemivariational problems by considering a new class of abstract problems based on the mathematical modeling of various physical phenomena.

It should be mentioned that Problem 1.1 is a generalized problem that contains the problems considered by Matei (2019, 2022) as special cases.

If $\eta(\cdot,z) = z,\ h(z,v) = f(v) - f(z),\ k(\lambda,\mu) = l(\mu) - l(\lambda),$ for all $z, v \in Z$ and $\lambda, \mu \in \Lambda$, then Problem 1.1 reduces to the following mixed variational-hemivariational problem: Given $f \in X$, find $(z,\lambda) \in Z \times \Lambda$ such that for all $(v,\mu) \in Z \times \Lambda$,

$$\begin{cases} A\big(z, v-z\big) + B(v-z,\lambda) + f(v) - f(z) + \Phi^0(Pz; Pv-Pz) \ge (f, v-z)_X \\ B(z,\mu-\lambda) - l(\mu) + l(\lambda) - \Upsilon^0(Q\lambda; Q\mu - Q\lambda) \le 0, \end{cases} \tag{1.1}$$

which was considered by Matei (2022).

If X, Ω are the real reflexive Banach spaces, $X = X_P$, $A(\cdot,\cdot) = (\cdot,\cdot)_{X^*,X}$ stand for the duality pairing, $P(z) = z,\ \eta(\cdot,z) = \eta(z),\ h \equiv 0, k \equiv 0, \Upsilon \equiv 0$ for all $z \in Z$, then Problem 1.1 reduces to the following mixed variational problem studied by Matei (2019).

Given $f \in X^*$, find $(z,\lambda) \in Z \times \Lambda$ such that for all $(v,\mu) \in Z \times \Lambda$,

$$\begin{cases} \big(\eta(z), v-z\big)_{X^*,X} + B(v-z,\lambda) + \Phi^0(z; v-z) \ge (f, v-z)_{X^*,X} \\ B(z,\mu-\lambda) \le 0. \end{cases} \tag{1.2}$$

The purpose of this paper is to study the existence and uniqueness of the results for Problem 1.1. Firstly, we establish the existence by using the technique involving the Tarafdar's fixed point theorem and some arguments in the nonsmooth analysis. Afterward, we discuss the uniqueness of the solution to Problem 1.1 under some strong monotonicity assumptions. Finally, our main results extend the corresponding results in Matei (2019, 2022).

For the reader's convenience, we identify some mathematical tools which will be required for the sequel (Migórski et al., 2013).

Let $\Phi : W \to \mathbb{R}$ be a locally Lipschitz functional on a Hilbert space W. The Clarke's generalized gradient of Φ at $z \in W$ is defined by

$$\partial\Phi(z) = \left\{ \zeta \in W^* \mid \Phi^0(z;v) \geq (\zeta, v)_{W^*,W} \quad \text{for all } v \in W \right\},$$

where $(\cdot,\cdot)_{W^*,W}$ stands for the duality pairing and $\Phi^0(z;v)$ stands for the Clarke's generalized derivative of Φ at $z \in W$ with respect to the direction $v \in W$, i.e.,

$$\Phi^0(z;v) = \limsup_{\substack{u \to z \\ t \downarrow 0}} \frac{\Phi(u+tv) - \Phi(z)}{t}.$$

As usual, W^* stands for the dual of W.

Proposition 1.2. *Let $\Phi : W \to \mathbb{R}$ be Lipschitz (of rank $k > 0$). Then*

(i) *$v \to \Phi^0(z;v)$ is finite, positively homogeneous, subadditive on W and satisfies*

$$\left|\Phi^0(z;v)\right| \leq k \|v\|_W .$$

(ii) *$(z,v) \to \Phi^0(z;v)$ is upper semicontinuous;*

(iii) *for each $z \in W, \Phi^0(z;v) = \max\left\{ (\zeta, v)_{W^*,W} \mid \zeta \in \partial\Phi(z) \right\}$ for each $v \in W$.*

The following theorem presents a fixed point result for set-valued mappings. We refer to Tarafdar (1987) for the proof of this theorem.

Theorem 1.3. *Let Ω be a Hilbert space and $\emptyset \neq \mathcal{K} \subset \Omega$ be a convex set. Let $H : \mathcal{K} \to 2^{\mathcal{K}}$ be a set-valued map such that*

(i) *for each $z \in \mathcal{K}$, $H(z)$ is a nonempty convex subset of $\mathcal{K}$;*

(ii) *for each $v \in \mathcal{K}$, $H^{-1}(v) = \{ z \in \mathcal{K} \mid v \in H(z) \}$ contains a relatively open subset $\mathcal{O}_v$ $(\mathcal{O}_v$ may be empty for some $v)$;*

(iii) $\bigcup_{v \in \mathcal{K}} \mathcal{O}_v = \mathcal{K}$;

(iv) *there exists a nonempty set* Π_0 *contained in a compact convex subset* Π_1 *of* $\mathcal{K}$ *such that* $\Sigma = \bigcap_{v \in \Xi_0} \left[\mathcal{O}_v\right]^c$ *is either empty or compact.*

Then, there exists $z^* \in \mathcal{K}$ *such that* $z^* \in H\left(z^*\right)$.

Notice that $2^{\mathcal{K}}$ denotes the family of all subsets of $\mathcal{K}$, and $\left[\mathcal{O}_v\right]^c$ is the complement of $\mathcal{O}_v$ in $\mathcal{K}$.

2. Main results

In this section, we investigate the existence and uniqueness of the solution for Problem 1.1. Throughout the paper, the symbol $\xrightarrow{w}$ (resp., $\to$) stands for the weak (resp., strong) convergence and $\gamma \in \Gamma$ and $f \in X$. To start our main results, we impose the following hypotheses on the data of Problem 1.1.

$(\mathbf{a}_1)$ $\left(X, (\cdot,\cdot)_X, \|\cdot\|_X\right)$ and $\left(\Omega, (\cdot,\cdot)_\Omega, \|\cdot\|_\Omega\right)$ are two Hilbert spaces.

$(\mathbf{a}_2)$ $Z \subset X$ and $\Lambda \subset \Omega$ are nonempty, bounded, closed, convex subsets.

$(\mathbf{a}_3)$ (i) The form $A : X \times X \to \mathbb{R}$ is bilinear and continuous, and there exists $m_A > 0$ such that

$$A\left(\eta(\cdot, z_2) - \eta(\cdot, z_1), z_2 - z_1\right) \geq m_A \left\|z_1 - z_2\right\|_X^2, \ \forall z_1, z_2 \in Z,$$

where $\eta : \Gamma \times X \to X$ is continuous and Γ is a metric space.

(ii) The form $B : X \times \Omega \to \mathbb{R}$ is bilinear and continuous.

$(\mathbf{a}_4)$ (i) The function $h : Z \times Z \to \mathbb{R}$ is upper semicontinuous in the first component, convex in the second component, and $h(z, z) = 0, \forall z \in Z.$

(ii) The function $k : \Lambda \times \Lambda \to \mathbb{R}$ is upper semicontinuous in the first component, convex in the second component, and $k(\lambda, \lambda) = 0, \forall \lambda \in \Lambda.$

$(\mathbf{a}_5)$ $P : X \to X_P$ and $Q : \Omega \to \Omega_Q$ are linear and compact operators $\left(X_P, (\cdot,\cdot)_{X_P}, \|\cdot\|_{X_P}\right)$ and $\left(\Omega_Q, (\cdot,\cdot)_{\Omega_Q}, \|\cdot\|_{\Omega_Q}\right)$ are Hilbert spaces.

$(\mathbf{a}_6)$ (i) The function $\Phi : X_P \to \mathbb{R}$ is Lipschitz continuous (of rank $L_\Phi > 0$).

(ii) The function $\Upsilon : \Omega_Q \to \mathbb{R}$ is Lipschitz continuous (of rank $L_\Upsilon > 0$).

$(\mathbf{a}_7)$ (i) There exists $m_h \geq 0$ such that

$$h(z_2, z_1) + h(z_1, z_2) \geq m_h \left\|z_1 - z_2\right\|_X^2, \ \forall z_1, z_2 \in X.$$

(ii) There exists $m_k \geq 0$ such that

$$k(\lambda_2,\lambda_1)+k(\lambda_1,\lambda_2)\geq m_k\left\|\lambda_1-\lambda_2\right\|_{\Omega}^2,\ \forall\lambda_1,\lambda_2\in\Omega.$$

$(\mathbf{a}_8)$ (i) There exists $m_\Phi\geq 0$ such that

$$\Phi^0(p_2;p_1-p_2)+\Phi^0(p_1;p_2-p_1)\leq m_\Phi\left\|p_1-p_2\right\|_{X_P}^2,\ \forall p_1,p_2\in X_P.$$

(ii) There exists $m_\Upsilon\geq 0$ such that

$$\Upsilon^0(q_2;q_1-q_2)+\Upsilon^0(q_1;q_2-q_1)\leq m_\Upsilon\left\|q_1-q_2\right\|_{\Omega_Q}^2,\ \forall q_1,q_2\in\Omega_Q.$$

Remark 2.1.

(i) The condition $(\mathbf{a}_7)$ is known as the strong monotonicity assumption of the equilibrium functions $h:Z\times Z\to\mathbb{R}$ and $k:\Lambda\times\Lambda\to\mathbb{R}$.

(ii) For locally Lipschitz functions $\Phi:X_P\to\mathbb{R}$ and $\Upsilon:\Omega_Q\to\mathbb{R}$, it follows from the definition of Clarke's generalized gradient and Proposition 1.2 that the inequalities in $(\mathbf{a}_8)$ are equivalent to

$$\left(w_1-w_2,p_1-p_2\right)_{X_P^*,X_P}\geq -m_\Phi\left\|p_1-p_2\right\|_{X_P}^2,\forall w_1\in\partial\Phi\left(p_1\right),w_2\in\partial\Phi\left(p_2\right),\forall p_1,p_2\in X_P,$$

$$\left(\zeta_1-\zeta_2,q_1-q_2\right)_{\Omega_Q^*,\Omega_Q}\geq -m_\Upsilon\left\|q_1-q_2\right\|_{\Omega_Q}^2,\forall\zeta_1\in\partial\Upsilon\left(q_1\right),\zeta_2\in\partial\Upsilon\left(q_2\right),\forall q_1,q_2\in\Omega_Q,$$

respectively which are known as the relaxed monotonicity conditions (Migórski et al.,2013). Next, we consider an auxiliary problem as follows:

Problem 2.2. Given $\gamma\in\Gamma$ and $f\in X$, find $(z,\lambda)\in Z\times\Lambda$ such that

$$\begin{aligned}&A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)\\&\qquad+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\geq(f,v-z)_X\end{aligned}$$

for all $(v,\mu)\in Z\times\Lambda$.

Lemma 2.3. $(z,\lambda)\in Z\times\Lambda$ *is a solution to Problem* 1.1 *if and only if it is a solution to Problem* 2.2.

Proof. If $(z,\lambda)\in Z\times\Lambda$ is a solution to Problem 1.1, then for all $(v,\mu)\in Z\times\Lambda$, we can write

$$\begin{cases}A\big(\eta(\gamma,z),v-z\big)+B(v-z,\lambda)+h(z,v)+\Phi^0(Pz;Pv-Pz)\geq(f,v-z)_X\\-B(z,\mu-\lambda)+k(\lambda,\mu)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\geq 0.\end{cases}$$

By summing the two inequalities above and using the bilinearity of B, we obtain

$$\begin{aligned}&A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)\\&\qquad+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\geq(f,v-z)_X.\end{aligned}$$

Hence, (z,λ) is a solution to Problem 2.2.

Conversely, if $(z,\lambda)\in Z\times\Lambda$ is a solution to Problem 2.2, then

$$A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)$$
$$+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\geq (f,v-z)_X$$

for all $(v,\mu)\in Z\times\Lambda$. We set in the inequality above with $\mu=\lambda$ and $v=z$ and get

$$\begin{cases} A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\lambda)+h(z,v)+k(\lambda,\lambda) \\ \qquad +\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;0)\geq (f,v-z)_X, \\ A\big(\eta(\gamma,z),0\big)+B(z,\lambda)-B(z,\mu)+h(z,z)+k(\lambda,\mu) \\ \qquad +\Phi^0(Pz;0)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\geq 0. \end{cases}$$

As $A\big(\eta(\gamma,z),0\big)=h(z,z)=k(\lambda,\lambda)=\Phi^0(Pz;0)=\Upsilon^0(Q\lambda;0)=0$ and the bifunction B is bilinear, it follows that

$$\begin{cases} A\big(\eta(\gamma,z),v-z\big)+B(v-z,\lambda)+h(z,v)+\Phi^0(Pz;Pv-Pz)\geq (f,v-z)_X \\ B(z,\mu-\lambda)-k(\lambda,\mu)-\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\leq 0, \end{cases}$$

i.e., (z,λ) is a solution to Problem 1.1. □

The following result provides the existence of solutions to Problem 1.1.

Theorem 2.4. *Suppose that* $(\mathbf{a}_1)-(\mathbf{a}_6)$ *hold. Then Problem* 1.1 *has at least one solution* $(z,\lambda)\in Z\times\Lambda$.

Proof. It follows from Lemma 2.3 that it is enough to prove that Problem 2.2 has at least one solution $(z,\lambda)\in Z\times\Lambda$. Arguing by contradiction, for each $(z,\lambda)\in Z\times\Lambda$, there exists $(v,\mu)\in Z\times\Lambda$ such that

$$A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)$$
$$+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)< (f,v-z)_X.$$

For each $\gamma\in\Gamma$, let us define a set-valued map $H_\gamma: Z\times\Lambda\to 2^{Z\times\Lambda}$ as follows:

$$H_\gamma(z,\lambda)=\left\{(v,\mu)\in Z\times\Lambda \;\middle|\; \begin{array}{l} A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu) \\ \qquad +\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)< (f,v-z)_X \end{array}\right\}.$$

Let $(z,\lambda)\in Z\times\Lambda$. As Problem 2.2 has no solution, then $H_\gamma(z,\lambda)\neq\emptyset$. Moreover, the set $H_\gamma(z,\lambda)$ is convex. Indeed, let $(v_1,\mu_1),(v_2,\mu_2)\in H_\gamma(z,\lambda)$ and $s\in[0,1]$. Since $(v_1,\mu_1),(v_2,\mu_2)\in Z\times\Lambda$ and $Z\times\Lambda$ is a convex set, $s(v_1,\mu_1)+(1-s)(v_2,\mu_2)\in Z\times\Lambda$. Set $v_s=sv_1+(1-s)v_2$ and $\mu_s=s\mu_1+(1-s)\mu_2$. It follows from $(v_1,\mu_1),(v_2,\mu_2)\in H_\gamma(z,\lambda)$ that

$$A\big(\eta(\gamma,z),v_1-z\big)+B(v_1,\lambda)-B(z,\mu_1)+h(z,v_1)+k(\lambda,\mu_1)$$
$$+\Phi^0(Pz;Pv_1-Pz)+\Upsilon^0(Q\lambda;Q\mu_1-Q\lambda)< (f,v_1-z)_X$$

and

$$A\big(\eta(\gamma,z),v_2-z\big)+B(v_2,\lambda)-B(z,\mu_2)+h(z,v_2)+k(\lambda,\mu_2)$$
$$+\Phi^0(Pz;Pv_2-Pz)+\Upsilon^0(Q\lambda;Q\mu_2-Q\lambda)<(f,v_2-z)_X.$$

As h,k,Φ^0 and Υ^0 are convex in the second argument, using the two inequalities above, we have

$$\begin{aligned}
&A\big(\eta(\gamma,z),v_s-z\big)+B(v_s,\lambda)-B(z,\mu_s)+h(z,v_s)+k(\lambda,\mu_s)\\
&\quad+\Phi^0(Pz;Pv_s-Pz)+\Upsilon^0(Q\lambda;Q\mu_s-Q\lambda)\\
&\le s\Big[A\big(\eta(\gamma,z),v_1-z\big)+B(v_1,\lambda)-B(z,\mu_1)+h(z,v_1)+k(\lambda,\mu_1)\\
&\quad+\Phi^0(Pz;Pv_1-Pz)+\Upsilon^0(Q\lambda;Q\mu_1-Q\lambda)\Big]\\
&\quad+(1\text{-}s)\Big[A\big(\eta(\gamma,z),v_2-z\big)+B(v_2,\lambda)-B(z,\mu_2)+h(z,v_2)+k(\lambda,\mu_2)\\
&\quad+\Phi^0(Pz;Pv_2-Pz)+\Upsilon^0(Q\lambda;Q\mu_2-Q\lambda)\Big]\\
&< s(f,v_1-z)_X+(1-s)(f,v_2-z)_X\\
&=\big(f,sv_1+(1-s)v_2-z\big)_X.
\end{aligned}$$

Thus,

$$s\big(v_1,\mu_1\big)+(1-s)\big(v_2,\mu_2\big)=\big(v_s,\mu_s\big)\in H_\gamma(z,\lambda)$$

and so (i) in Theorem 1.3 is verified.

We now introduce $H_\gamma^{-1}(v,\mu)$ and $\mathcal{O}^{\gamma}_{(v,\mu)}$ as follows:

$$H_\gamma^{-1}(v,\mu)=\big\{(z,\lambda)\in Z\times\Lambda \mid (v,\mu)\in H_\gamma(z,\lambda)\big\}$$

$$\mathcal{O}^{\gamma}_{(v,\mu)}=\left\{(z,\lambda)\in Z\times\Lambda \;\middle|\; \begin{array}{l} A\big(\eta(\gamma,v),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)\\ +\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)<(f,v-z)_X+m_A\|v-z\|_X^2\end{array}\right\}.$$

Let $(v,\mu)\in Z\times\Lambda$. Then,

$$\big[H_\gamma^{-1}(v,\mu)\big]^c\subset\big[\mathcal{O}^{\gamma}_{(v,\mu)}\big]^c. \tag{2.2}$$

Indeed, since

$$H_\gamma^{-1}(v,\mu)=\left\{(z,\lambda)\in Z\times\Lambda \;\middle|\; \begin{array}{l} A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)\\ +\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)<(f,v-z)_X\end{array}\right\},$$

if $(z,\lambda)\in\big[H_\gamma^{-1}(v,\mu)\big]^c$, then

$$A\big(\eta(\gamma,z),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)$$
$$+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\ge(f,v-z)_X.$$

By using $(\mathbf{a}_2)$, we obtain

$$A\big(\eta(\gamma,v),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)$$
$$+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)\ge(f,v-z)_X+m_A\|v-z\|_X^2 .$$

This implies that $(z,\lambda)\in\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$, and so (2.2) holds. Thus,

$$\mathcal{O}^{\gamma}_{(v,\mu)}\subset H_{\gamma}^{-1}(v,\mu).$$

The set $\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$ is weakly closed. Indeed, let $(z_m,\lambda_m)_m\subset\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$ be a sequence such that $(z_m,\lambda_m)\xrightarrow{w}(z,\lambda)$ in $X\times\Omega$ as $m\to\infty$. As, for all $m\ge 1$, one has

$$\begin{aligned}&A\big(\eta(\gamma,v),v-z_m\big)+B(v,\lambda_m)-B(z_m,\mu)+h(z_m,v)+k(\lambda_m,\mu)\\&+\Phi^0(Pz_m;Pv-Pz_m)+\Upsilon^0(Q\lambda_m;Q\mu-Q\lambda_m)\ge(f,v-z_m)_X+m_A\|v-z_m\|_X^2\end{aligned}\qquad(2.3)$$

then, using the assumptions supposed in this theorem and Proposition 1.2, as $z_m\xrightarrow{w}z$ in X as $m\to\infty$ and $\lambda_m\xrightarrow{w}\lambda$ in Ω as $m\to\infty$, by passing to the superior limit as $m\to\infty$ in (2.3), we have

$$A\big(\eta(\gamma,v),v-z\big)+B(v,\lambda)-B(z,\mu)+h(z,v)+k(\lambda,\mu)$$
$$+\Phi^0(Pz;Pv-Pz)+\Upsilon^0(Q\lambda;Q\mu-Q\lambda)<(f,v-z)_X+m_A\|v-z\|_X^2 .$$

Hence, we deduce that $(z,\lambda)\in\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$. As $\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$ is weakly closed then $\mathcal{O}^{\gamma}_{(v,\mu)}$ is weakly open. As a result, $\mathcal{O}^{\gamma}_{(v,\mu)}$ is a relatively open subset in the weak topology. Thus, the assumption (ii) of Theorem 1.3 holds.

Next, we prove that the equality $Z\times\Lambda=\bigcup_{(v,\mu)\in Z\times\Lambda}\mathcal{O}^{\gamma}_{(v,\mu)}$. It can be easily seen that $\bigcup_{(v,\mu)\in Z\times\Lambda}\mathcal{O}^{\gamma}_{(v,\mu)}\subset Z\times\Lambda$.

We verify that

$$Z\times\Lambda\subset\bigcup_{(v,\mu)\in Z\times\Lambda}\mathcal{O}^{\gamma}_{(v,\mu)}.$$

Indeed, let $(z,\lambda)\in Z\times\Lambda$. Since Problem 2.2 has no solution, we deduce that there exists $(v,\mu)\in Z\times\Lambda$ such that $(z,\lambda)\in\mathcal{O}^{\gamma}_{(v,\mu)}$. We conclude that the condition (iii) of Theorem 1.3 holds true.

To show the condition (iv) of Theorem 1.3, we denote $\Pi_0=\Pi_1=Z\times\Lambda$. As for each $(v,\mu)\in Z\times\Lambda$, $\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$ is a weakly closed set, the intersection

$$\Sigma=\bigcap_{(v,\mu)\in Z\times\Lambda}\left[\mathcal{O}^{\gamma}_{(v,\mu)}\right]^c$$

is an empty or weakly closed set. Since the subset $Z\times\Lambda$ is a nonempty closed convex bounded on the reflexive Banach space $X\times\Omega$, it follows that $Z\times\Lambda$ is weakly compact. Hence, Σ is either empty or weakly compact.

Thus, all hypotheses of Theorem 1.3 hold true in the weak topology on $X\times\Omega$. We deduce that there exists $\left(z^*,\lambda^*\right)\in H_\gamma\left(z^*,\lambda^*\right)$, i.e.,

$$A\left(\eta(\gamma,z^*),z^*-z^*\right)+B(z^*,\lambda^*)-B(z^*,\lambda^*)+h(z^*,z^*)+k(\lambda^*,\lambda^*)$$
$$+\Phi^0(Pz^*;Pz^*-Pz^*)+\Upsilon^0(Q\lambda^*;Q\lambda^*-Q\lambda^*)<(f,z^*-z^*)_X. \qquad (2.4)$$

Since

$$A\left(\eta(\gamma,z^*),0\right)=h(z^*,z^*)=k(\lambda^*,\lambda^*)=\Phi^0(Pz^*;0)=\Upsilon^0(Q\lambda^*;0)=0,$$

the inequality (2.4) is impossible.

Therefore, Problem 2.2 has at least one solution. Then, Problem 1.1 has at least one solution. This completes the proof. □

Remark 2.5. In special cases of the problems (1.1) and (1.2) in Section 1, Theorem 2.4 extends Theorem 2 in Matei (2022) and Theorem 2 in Matei (2019) to the mixed variational-hemivariational problem provided by the perturbed parameter λ in the operator η and equilibrium functions h,k . Therefore, this class of problems is useful for investigating well-posedness by perturbations and Hölder continuity of solution mapping.

Theorem 2.6. *Assume that the assumptions* $(\mathbf{a}_1)-(\mathbf{a}_8)$ *are fulfilled and*

$$\min\left\{m_A+m_h-m_\Phi\left\|P\right\|^2_{L(X,X_P)},m_k-m_\Upsilon\left\|Q\right\|^2_{L(\Omega,\Omega_Q)}\right\}>0. \qquad (2.4)$$

Then, Problem 1.1 *has a unique solution.*

Proof. If followed from Theorem 2.4 that Problem 1.1 has at least one solution. We need to prove our uniqueness. Indeed, let $(z_j,\lambda_j)\in Z\times\Lambda,\ (j\in\{1,2\})$, be two solutions to Problem 1.1. Then, for all $(v,\mu)\in Z\times\Lambda$, we get

$$\begin{cases}A\left(\eta(\gamma,z_j),v-z_j\right)+B(v-z_j,\lambda)+h(z_j,v)+\Phi^0(Pz_j;Pv-Pz_j)\geq(f,v-z_j)_X\\ B(z_j,\mu-\lambda_j)-k(\lambda_j,\mu)-\Upsilon^0(Q\lambda_j;Q\mu-Q\lambda_j)\leq 0,\end{cases} \qquad (2.5)$$

Let $v=z_2,\mu=\lambda_2$ if $j=1$ and $v=z_1,\mu=\lambda_1$ if $j=2$ in (2.5). By Lemma 2.2, we can write

$$A\left(\eta(\gamma,z_2),z_1-z_2\right)+B(z_1,\lambda_2)-B(z_2,\lambda_1)+h(z_2,z_1)+k(\lambda_2,\lambda_1)$$
$$+\Phi^0(Pz_2;Pz_1-Pz_2)+\Upsilon^0(Q\lambda_2;Q\lambda_1-Q\lambda_2)\geq(f,z_1-z_2)_X,$$
$$A\left(\eta(\gamma,z_1),z_2-z_1\right)+B(z_2,\lambda_1)-B(z_1,\lambda_2)+h(z_1,z_2)+k(\lambda_1,\lambda_2)$$
$$+\Phi^0(Pz_1;Pz_2-Pz_1)+\Upsilon^0(Q\lambda_1;Q\lambda_2-Q\lambda_1)\geq(f,z_2-z_1)_X.$$

We sum up the above inequalities to obtain

$$\begin{aligned}&A\big(\eta(\gamma,z_2),z_1-z_2\big)+A\big(\eta(\gamma,z_1),z_2-z_1\big)\\&\qquad\qquad+h(z_2,z_1)+h(z_1,z_2)+k(\lambda_2,\lambda_1)+k(\lambda_1,\lambda_2)\\&\geq-\Phi^0(Pz_2;Pz_1-Pz_2)-\Phi^0(Pz_1;Pz_2-Pz_1)\\&\qquad\qquad-\Upsilon^0(Q\lambda_2;Q\lambda_1-Q\lambda_2)-\Upsilon^0(Q\lambda_1;Q\lambda_2-Q\lambda_1)\end{aligned}\tag{2.6}$$

By the conditions $(\mathbf{a}_3)(1)$ and $(\mathbf{a}_7)$, we have

$$\begin{aligned}&A\big(\eta(\gamma,z_2),z_1-z_2\big)+A\big(\eta(\gamma,z_1),z_2-z_1\big)\\&\qquad\qquad+h(z_2,z_1)+h(z_1,z_2)+k(\lambda_2,\lambda_1)+k(\lambda_1,\lambda_2)\\&\leq-\big(m_A+m_h\big)\|z_1-z_2\|_X^2-m_k\|\lambda_1-\lambda_2\|_\Omega^2.\end{aligned}\tag{2.7}$$

The condition $(\mathbf{a}_8)$ verifies that

$$\begin{aligned}&-\Phi^0(Pz_2;Pz_1-Pz_2)-\Phi^0(Pz_1;Pz_2-Pz_1)\\&\qquad\qquad-\Upsilon^0(Q\lambda_2;Q\lambda_1-Q\lambda_2)-\Upsilon^0(Q\lambda_1;Q\lambda_2-Q\lambda_1)\\&\geq-m_\Phi\|P\|_{L(X,X_P)}^2\|z_1-z_2\|_X^2-m_\Upsilon\|Q\|_{L(\Omega,\Omega_Q)}^2\|\lambda_1-\lambda_2\|_\Omega^2.\end{aligned}\tag{2.8}$$

Having in mind relations (2.6)-(2.8), it follows that

$$\left(m_A+m_h-m_\Phi\|P\|_{L(X,X_P)}^2\right)\|z_1-z_2\|_X^2+\left(m_k-m_\Upsilon\|Q\|_{L(\Omega,\Omega_Q)}^2\right)\|\lambda_1-\lambda_2\|_\Omega^2\leq 0.$$

Hence,

$$\Delta\left(\|z_1-z_2\|_X^2+\|\lambda_1-\lambda_2\|_\Omega^2\right)\leq 0,\tag{2.9}$$

where

$$\Delta=\min\left\{m_A+m_h-m_\Phi\|P\|_{L(X,X_P)}^2,\ m_k-m_\Upsilon\|Q\|_{L(\Omega,\Omega_Q)}^2\right\}.$$

By the condition (2.4), $\Delta>0$, hence it follows from (2.9) that $z_1=z_2$ and $\lambda_1=\lambda_2$. Therefore, Problem 1.1 has a unique solution $(z,\lambda)\in Z\times\Lambda$. □

Remark 2.7. In Theorem 2.6, we derived a new condition (2.4) to establish the uniqueness of a solution to Problem 1.1 without the assumption $\Upsilon^0\equiv 0$. Thus, this result is a significant extension of Theorem 7 in Matei (2022) in the case that Λ is bounded.

3. Conclusion

In this paper, we have introduced a general kind of mixed parametric variational-hemivariational problems involving the Clarke's generalized derivatives and equilibrium functions (Problem 1.1). Using a fixed point result for set-valued mappings and the arguments of monotonicity, we establish the results concerning the existence and uniqueness of a solution to Problem 1.1. Our main results extend to the corresponding results by Matei (2019, 2022).

As future research, we intend to study well-posedness by perturbations, error bounds, Hölder continuity of solution mapping, and applications to contact mechanics for Problem 1.1.

- ❖ ***Conflict of Interest:*** *Author have no conflict of interest to declare.*
- ❖ ***Acknowledgments:*** *This work is supported by the Ministry of Education and Training of Vietnam under Grant No. B2021.SPD.03.*

REFERENCES

Bai, Y., Migórski, S., & Zeng, S.D. (2020). A class of generalized mixed variational-hemivariational inequalities I: Existence and uniqueness results. *Comput Math Appl*, *79*(10), 2897-2911.

Han, W., Migórski, S., & Sofonea, M. (2014). A class of variational–hemivariational inequalities with applications to frictional contact problems. *SIAM J Math Anal*, *46*, 3891-3912.

Matei, A. (2022). On a new class of abstract mixed variational-hemivariational problems. *Commun Nonlinear Sci Numer Simul*, *104*, 106046.

Matei, A. (2019). A mixed hemivariational-variational problem and applications. *Comput Math Appl*, *77*(11), 2989-3000.

Migórski, S., Ochal, A., & Sofonea, M. (2013). Nonlinear Inclusions and Hemivariational Inequalities Models and Analysis of Contact Problems. In *Advances in Mechanics and Mathematics*, vol. 26. Springer, New York.

Migórski, S., Bai, Y., & Zeng, S.D. (2019). A class of generalized mixed variational-hemivariational inequalities II: Applications. *Nonlinear Anal RWA, 50*, 633-50.

Nguyen, V. H., Migórski, S., Vo, M. T., V.M., & Zeng, S. D. (2020). Gap functions and error bounds for variational–hemivariational inequalitie. *Acta Appl Math, 169,* 691-709.

Nguyen, V. H., Vo, M. T., Liu, Z., & Yao, J.C. (2021). A novel approach to Hölder continuity of a class of parametric variational-hemivariational inequalities. *Oper Res Lett, 49,* 283-289.

Nguyen, V. H., Vo, M. T., & Pitea, A. (2020). Global error bounds for mixed Quasi-Hemivariational inequality problems on Hadamard manifolds. *Optimization, 69,* 2033-2052.

Panagiotopoulos, P.D. (1985). Nonconvex problems of semipermeable media and related topics. *Z Angew Math Mech, 65*, 29-36

Panagiotopoulos, P.D. (1993). Hemivariational Inequalities. In *Applications in Mechanics and Engineering*, Springer, Berlin.

Sofonea, M., & Danan, D. (2018). Numerical analysis of stationary variational-hemivariational inequalities. *Numer Math*, *139*, 563-592.

Tarafdar, E. (1987). A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem. *J Math Anal Appl, 128*(2), 475-479.

, M. T. (2022). Upper-bound error estimates for double phase obstacle problems with clarke's subdifferential. *Numer Funct Anal Optim, 43*(4), 463-485.

KẾT QUẢ TỒN TẠI VÀ DUY NHẤT NGHIỆM CHO BÀI TOÁN BIẾN PHÂN-NỬA BIẾN PHÂN HỖN HỢP THAM SỐ

Võ Minh Tâm

Trường Đại học Đồng Tháp, Việt Nam

Tác giả liên hệ: Võ Minh Tâm – Email: vmtam@dthu.edu.vn

Ngày nhận bài: 21-8-2022; ngày nhận bài sửa:17-10-2022; ngày duyệt đăng: 31-10-2022

TÓM TẮT

Trong bài báo này, chúng tôi giới thiệu một lớp tổng quát các bài toán biến phân-nửa biến phân hỗn hợp tham số liên quan đến đạo hàm suy rộng Clarke và những hàm cân bằng (viết tắt là PMVHP). Dựa trên kĩ thuật chứng minh liên quan đến định lí điểm bất động Tarafdar và một số tính *chất của giải tích phi tuyến, chúng tôi nghiên cứu sự tồn tại nghiệm của bài toán PMVHP. Hơn nữa, chúng tôi cũng thiết lập được sự duy nhất nghiệm đến bài toán PMVHP dưới một số giả thiết đơn điệu mạnh. Các kết quả chính của chúng tôi là mở rộng những kết quả tương ứng trong các công trình Matei (2019, 2022).*

Từ khóa: đạo hàm suy rộng Clarke; tồn tại và duy nhất nghiệm; định lí điểm bất động cho ánh xạ đa trị; bài toán biến phân-nửa biến phân hỗn hợp tham số