TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 20, Số 3 (2023): 491-504
Website: https://journal.hcmue.edu.vn

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 20, No. 3 (2023): 491-504
https://doi.org/10.54607/hcmue.js.20.3.3730(2023)

ISSN: 2734-9918

Research Article

PROXIMAL POINT ALGORITHM FOR THE GENERALIZED P_0 VARIATIONAL INEQUALITIES PROBLEMS

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Received: February 20, 2023; Revised: March 23, 2023; Accepted: March 24, 2023

ABSTRACT

This paper studies the proximal point algorithm for the class of generalized P_0 variational inequalities. By using the upper semicontinuity result establishing the class of weakly univalent operators, we show that the iterative sequence generated by the algorithm is bounded, approaches to the solution set of the initial problem, and each of its accumulation points is a solution to the problem, provided that the solution set is bounded. We also give an example to show the necessity of boundedness.

Keywords: convergence; natural mappings; P_0- functions; *P*- functions; proximal point algorithm; univalent operators; variational inequalities

Mathematics Subject Classification (2010) 47H05, 47J20, 47J25, 49J40

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1. Introduction

The variational inequalities (VIs) have many applications in different realistic models, such as in engineering and economics (Facchinei & Pang, 2003), and contains many classes of problems, such as complementarity problems, a system of equations problems, fixed point problems, and Nash equilibrium problems (Facchinei & Pang, 2003; Kinderleher & Stampacchia, 1980).

Many different methods for solving VIs were proposed (Facchinei & Pang, 2003). Among them are two approaches based on the regularization idea, namely the Tikhonov regularization method (TRM) and the proximal point algorithm (PPA). Those two algorithms, which are crucial for solving monotone problems (Facchinei & Pang, 2003), are expected to be effective when applied to non-monotone problems. Some investigations in this direction have been done. For example, the convergence theorems for the Tikhonov

Cite this article as: Tran Hong Mo, Pham Duy Khanh, & Huynh Phuoc Toan (2023**)**. Proximal point algorithm for the generalized P_0 variational inequalities problems. *Ho Chi Minh City University of Education Journal of Science, 20*(3), 491-504.

regularization method applied to finite-dimensional monotone and pseudo monotone problems could be found in Facchinei and Pang (2003) and Tam, Yao, and Yen (2008), and Nguyen (2006), respectively. For the class of problems more significant than the monotone ones, Facchinei and Pang (1998) discussed the application of TRM to generalized P_0 problems and established the convergence results for the class of subanalytic generalized P_0 operator problems. Considering the PPA, Martinet (1970) proposed the exact method, and Rockafellar (1976) suggested and applied the inexact version for a class of monotone VIs. In addition, Noor (2002) used the proximal point method to solve the pseudomotone VIs and obtained some convergence theorems. For the non-monotone problems, Yamashita (Yamashita & Fukushima, 2001) applied this algorithm to the P_0 complementarity problem and constructed several algorithms to solve the original problem. The convergence theorem for the general class of generalized P_0 problems when applied both methods is still an open question.

In this paper, we apply the PPA to the class of generalized P_0 problems and examine the behaviors of the sequence of solutions generated by this algorithm. We prove that the iterative sequence generated by the PPA approaches for the solution set of the original problem, given that the solution set is bounded. As a consequence, this sequence is bounded, and all of its accumulation points are solutions to the problem. This result has already been established for the sequence of solutions generated by the TRM when applied to the complementarity P_0 problems in Facchinei and Kanzow (1999). We also provide an example to show that the boundedness cannot be dropped. In addition, some new convergence of the PPA for the generalized P_0 problem will be obtained. Our proof is based on the upper semicontinuity results for the class of weakly univalent operators (Ravindran & Gowda, 2000).

The rest of the paper is organized as follows. In the next section, we formally define the concept of variational inequalities and summarize some primary results that are needed for the main theorems. Section 3 presents the application of the PPA to the class of VIs of the generalized P_0 type and obtains the main results. Finally, section 4 contains some remarks and open questions for future studies.

2. Preliminaries

This section considers a nonempty subset K of $\mathbb{R}^n$ and a mapping F from $\mathbb{R}^n$ to $\mathbb{R}^n$. We also define variational inequality problem given by K and F , as well as some mathematical tools used to establish the main results in the next section.

2.1. Generalized P_0 variational inequalities problems

Definition 2.1.1. The variational *inequality problem* defined by K and F, denoted by $\mathrm{VI}(K,F)$, is to find a vector $x^* \in K$ such that

$$\left\langle F\left(x^*\right), x - x^*\right\rangle \geq 0, \quad \forall x \in K. \tag{1}$$

The set of solutions to this problem is denoted by $\mathrm{SOL}(K,F)$.

We concentrate on problems attaching with classes of P_0, P operators, which include a class of (strict) monotone operators.

Definition 2.1.2. (More & Rheinboldt, 1973) The mapping $F = \left(F_1, F_2, ..., F_n\right): \mathbb{R}^n \to \mathbb{R}^n$ is called

(i) P_0 – function (in the classical sense) on K if for any pair of distinct vectors x, y in K, there exists an index $k = k(x,y) \in \{1, 2, ..., n\}$ such that

$$x_k \neq y_k \text{ and } \left(x_k - y_k\right)\left[F_k(x) - F_k(y)\right] \geq 0;$$

(ii) P – function (in classical sense) on K if for any pair of distinct vectors x, y in K, we have that

$$\max_{1 \leq k \leq n}\left(x_k - y_k\right)\left[F_k(x) - F_k(y)\right] > 0;$$

We next extend the definitions for the P_0 and P operators when K has a special structure, namely Cartesian structure. A subset K of $\mathbb{R}^n$ is called to have the Cartesian structure if it can be written as

$$K = \prod_{j=1}^{m} K^j \tag{2}$$

where each K^j is a nonempty subset of $\mathbb{R}^{n_j}$ with $\sum_{j=1}^{m} n_j = n.$ Correspondingly, we also partition and represent the vector x in $\mathbb{R}^n$ and operator F in the following way:

$$x = \left(x^1, x^2, \cdots, x^m\right) \text{ and } F(x) = \left(F^1(x), F^2(x), \cdots, F^m(x)\right),$$

where each x^j and $F^j(x)$ belong to $\mathbb{R}^{n_j}$ for all index j in $\{1, \cdots, m\}$.

Definition 2.1.3. (Facchinei & Pang, 1998) Let K be a set of which structure is given by (2).

(a) F is a generalized P_0 – function with respect to K if for any pair of distinct vectors x and y in K, there exists an index $j_0 \in \{1, 2, ..., m\}$ such that

$$x^{j_0} \neq y^{j_0} \text{ and } \left\langle x^{j_0} - y^{j_0}, F^{j_0}(x) - F^{j_0}(y)\right\rangle \geq 0.$$

(b) F is a generalized P –function with respect to K if for any pair of distinct vectors x and y in K, we have that

$$\max_{1\le j\le m}\left\langle x^j - y^j, F^j(x) - F^j(y)\right\rangle > 0.$$

Clearly, if F is a (strict) monotone operator on set K given by (2), then F is also a generalized $(P-)$ P_0- function with respect to K.

The VI problem, whose defining set K is given by the Cartesian product, is called the partitioned VI. The partitioned $\text{VI}(K,F)$ where F is a generalized $(P-)$ P_0-function with respect to K is called a generalized $(P)\,P_0$ problem. The classes of generalized P_0 (P) problems include some interesting cases.

- If $m = n$ (so that $n_j = 1$ for all j) and $K^j = \mathbb{R}_+$, the $\text{VI}(K,F)$) reduces to a nonlinear complementarity problem (Facchinei & Pang, 2003) with $P_0-(P-)$ function in the classical sense.
- If $m = 1$, so that $n_1 = n,$ F is a generalized $P_0-(P-)$ function on K if and only if F is monotone (strict monotone) on K.
- If $m = n$ and $K^j = \left[a^j, b^j\right]$, the problem becomes the box constrained VI (Ravindran & Gowda, 2000) and F is a generalized P_0- function on K if and only if F is a P_0- function in the classical sense. If $K^j = \mathbb{R}$ then F is a generalized P_0- $(P-)$ function on K if and only if F is a $P_0-(P-)$ function in the classical sense and the $\text{VI}(K,F)$ reduces to the system of equations $F(x) = 0$.

2.2. Natural map associated with the VI problem

The natural map has a close relationship with the variational inequality problem and is used in many proofs of existing solutions to the VI (Facchinei & Pang, 2003) and in the analysis of sensitivity and stability (Facchinei & Pang, 2003). This mapping is constructed through the projection operator.

Proposition 2.2.1. **(**Kinderleher & Stampacchia, 1980) Let K be a nonempty, closed convex subset of $\mathbb{R}^n$. Then, for any vector x in $\mathbb{R}^n$, there exists a unique element y in K such that

$$\|x-y\| \le \|x-u\|,\ \forall u \in K. \tag{3}$$

The unique vector $y \in K$ satisfying (3) is called a projection of x onto K, denoted by $P_K(x)$. The mapping $P_K : \mathbb{R}^n \to K$ defined by $P_K(x) = y$ with y is the projection of x onto K is called the projection operator.

We then recall some well-known properties of the projection operator.

Proposition 2.2.2. Let K be a nonempty, closed convex subset of $\mathbb{R}^n$. Then $P_K(\cdot)$ is nonexpansive, that is

$$\|P_K(x) - P_K(y)\| \le \|x-y\|,\ \forall x, y \in \mathbb{R}^n. \tag{4}$$

Definition 2.2.1. Given a nonempty, closed convex subset K of $\mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$. The mapping $F_K^{\text{nat}} : K \to \mathbb{R}^n$ defined by

$$F_K^{\text{nat}}(v) = v - P_K(v - F(v)), \text{ with } v \in K$$

is called the natural map associated with the pair (K, F).

We can characterize the set of solutions to the VI problem through the zero set of the natural map.

Theorem 2.2.1. (Facchinei & Pang, 2003) Let K be a nonempty, closed convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$. Then, a vector x^* is a solution to the VI (K, F) problem if and only if x^* belongs to the zero set of F_K^{nat}.

2.3. Univalent and weakly univalent operator

We next introduce the concept of a weakly univalent operator, which has many useful properties in the analysis of the stability of solutions to the VI problem (Ravindran & Gowda, 2000; Sznajder & Gowda, 1999).

Definition 2.3.1. We say that $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is univalent if it is continuous and injective, and weakly univalent if there exist univalent functions $g_k : D \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $g_k \to g$ uniformly on every bounded subset of D.

An example of a weakly univalent operator is the natural map associated with the generalized P_0 VI problem.

Lemma 2.3.1. (Facchinei & Pang, 1998) Let $\text{VI}(K, F)$ be a generalized P_0 problem where F is a continuous mapping. Then the natural map F_K^{nat} associated with the pair (K, F) is a weakly univalent operator.

The following result describes an upper semicontinuity property of the inverse of a weakly univalent operator.

Theorem 2.3.1. (Ravindran & Gowda, 2000) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be weakly univalent and suppose that for a $q^* \in \mathbb{R}^n$,

$g^{-1}\left(q^*\right)$ is nonempty and compact.

Then for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every weakly univalent function h and for every vector q with

$$\sup_{\bar{\Omega}} \|h(x) - g(x)\| < \delta, \quad \|q - q^*\| < \delta$$

we have

$$\varnothing \neq h^{-1}(q) \subseteq g^{-1}\left(q^*\right) + \varepsilon B(1)$$

where $\Omega := g^{-1}\left(q^*\right) + \varepsilon B(1)$ and $B(1)$ denotes the open unit ball in $\mathbb{R}^n$. Moreover, $h^{-1}(q)$ and $g^{-1}(q)$ are nonempty, connected, and uniformly bounded for q in a neighborhood of q^*.

3. Proximal Point Algorithm for the generalized P_0 VI problem

The proximal point algorithm used to solve the variational inequality problems is proposed by Martinet (1970) and further studied by Rockafellar (1976). It is a popular method and often used for solving a class of monotone VI problems, and for a class of pseudomonotone ones (El Farouq, 2001; Noor, 2002; Rockafellar, 1976; Tam, Yao & Yen, 2008). The idea of this method is to substitute the original problem with a sequence of auxiliary problems that are, in some sense, better behaved.

The proximal point algorithm: *Choose a point x_0 in $\mathbb{R}^n$ and a sequence $\{\rho_k\}$ of positive numbers. If $x_{k-1} (k = 1, 2, \ldots)$ has been defined, then one can choose as x_k any solution of the problem $\mathrm{VI}\left(K, F^{(k)}\right)$ where*

$$F^{(k)}(x) := \rho_k F(x) + x - x_{k-1}, x \in \mathbb{R}^n, \tag{5}$$

that is $x_k \in K$ and

$$\left\langle \rho_k F\left(x_k\right) + x_k - x_{k-1}, y - x_k \right\rangle \geq 0, \forall y \in K.$$

To terminate the computation process after a finite number of steps and obtain the approximate solution of $\mathrm{VI}(K, F)$, one has to introduce a stopping criterion. (For example, one can terminate the computation when $\|x_k - x_{k-1}\| \leq \theta$, where $\theta > 0$ is a constant.)

First, we establish the solvability of the perturbed problems $\mathrm{VI}\left(K, F^{(k)}\right)$ where K is given by (2) and F is a generalized P_0-function with respect to K when implied the PPA to the original problem. In order to establish the result, we need the following lemma.

Lemma 3.1. (Facchinei & Pang, 2003) Let K be a subset of $\mathbb{R}^n$ given by (2), where each K^j is a closed convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generalized P_0-function with respect to K. Then, for every $\varepsilon > 0$ the $\mathrm{VI}\left(K, F_\varepsilon\right)$ problem has a unique solution where $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F_\varepsilon(x) = F(x) + \varepsilon x, \quad x \in \mathbb{R}^n.$$

Theorem 3.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generalized P_0-function and continuous on $\mathbb{R}^n$ and K be given by (2), with each K^j is a closed convex set. Then, for any $k \in \mathbb{N}$, $x_{k-1} \in \mathbb{R}^n$, the $\mathrm{VI}\left(K, F^{(k)}\right)$ has a unique solution.

Proof. For each natural number k, since F is a generalized P_0-function with respect to K so the mapping $G^{(k)}(\cdot) = \rho_k F(\cdot) - x_{k-1}$ is also a generalized P_0-function on K. Then the mapping $G_1^{(k)}$ defined by

$$G_1^{(k)}(x) = G^{(k)}(x) + x, \quad x \in K,$$

is a generalized P-function on K. This mapping is the mapping $F^{(k)}$ determined by the proximal point algorithm. By applying Lemma 3.1 with $\varepsilon = 1$, we have that the $\mathrm{VI}\left(K, G_1^{(k)}\right)$ problem has unique solution, which leads to the existence and uniqueness of solution to the $\mathrm{VI}\left(K, F^{(k)}\right)$.

We next examine some properties of the sequence of solutions $\{x_k\}$ generated by the auxiliary problems. We will see that $\{x_k\}$ approaches to the solution set $\mathrm{SOL}(K, F)$ under some specific conditions of the sequence $\{\rho_k\}$.

Theorem 3.2. Let $\mathrm{VI}(K, F)$ be a generalized P_0 problem where F is continuous on $\mathbb{R}^n$ and assume further that the solution set $\mathrm{S} := \mathrm{SOL}(K, F)$ is nonempty and bounded. Suppose that the sequence $\{\rho_k\}$ of positive numbers arising from the proximal point algorithm satisfies

$$\rho_k \to +\infty \text{ and } \frac{\|x_{k-1}\|}{\rho_k} \to 0 \text{ as } k \to \infty \tag{6}$$

where $\{x_k\}$ is the iteration generated by the proximal point algorithm, we have that

$$\lim_{k \to +\infty} \mathrm{dist}\left(x_k \mid \mathrm{S}\right) = 0.$$

Furthermore, the sequence $\{x_k\}$ is bounded, and each of its accumulation points is a solution to the original problem.

Proof. By Theorem 2.2.1, we have that

$$\mathrm{S} = \left(F_K^{\mathrm{nat}}\right)^{-1}(0)$$

Moreover, S is compact and nonempty. Therefore, by Theorem 2.3.1, we deduce that for any $\varepsilon > 0$ there exists a positive number δ such that for every weakly univalent mapping h satisfying

$$\sup_{\bar{\Omega}_\varepsilon} \left\| h(x) - F_K^{\mathrm{nat}}(x) \right\| < \delta, \tag{7}$$

we have

$$\varnothing \neq h^{-1}(0) \subseteq \left(F_K^{\mathrm{nat}}\right)^{-1}(0) + \varepsilon B(1) \tag{8}$$

where $\Omega_\varepsilon := \left(F_K^{\mathrm{nat}}\right)^{-1}(0) + \varepsilon B(1)$. We next show that the mapping $h = \tilde{F}_{k,K}^{\mathrm{nat}}$ where $\tilde{F}_{k,K}^{\mathrm{nat}}$ is the natural map associated with the $\mathrm{VI}\left(K, \tilde{F}^{(k)}\right)$ with the mapping $\tilde{F}^{(k)}$ defined by

$$\tilde{F}^{(k)}(x) = F(x) + \frac{1}{\rho_k}\left(x - x_{k-1}\right),$$

satisfies the condition (7) for sufficiently large positive integer *k*. It is easy to check that each $\tilde{F}^{(k)}$ is a generalized P-function with respect to K, therefore, by Lemma 2.3.1, it follows that $\tilde{F}_{k,K}^{\text{nat}}$ is weakly univalent. From the non-expansiveness property of the projection operator, for every $k \in \mathbb{N}$, we have

$$\left\|\tilde{F}_{k,K}^{\text{nat}}(x) - F_K^{\text{nat}}(x)\right\| \le \frac{1}{\rho_k}\left\|x - x_{k-1}\right\|, \forall x \in \bar{\Omega}_\varepsilon.$$

Let M be the radius of the open sphere containing $\left(F_K^{\text{nat}}\right)^{-1}(0)$, since $\left(F_K^{\text{nat}}\right)^{-1}(0)$ is compact, it follows that

$$\bar{\Omega}_\varepsilon = \overline{\left(F_K^{\text{nat}}\right)^{-1}(0) + \varepsilon B(1)} = \overline{\left(F_K^{\text{nat}}\right)^{-1}(0)} + \varepsilon \bar{B}(1) = \left(F_K^{\text{nat}}\right)^{-1}(0) + \varepsilon \bar{B}(1),$$

that is $\|x\| \le M + \varepsilon$ for every x in $\bar{\Omega}_\varepsilon$, hence, for each $k \in \mathbb{N}$, we have

$$\left\|\tilde{F}_{k,K}^{\text{nat}}(x) - F_K^{\text{nat}}(x)\right\| \le \frac{M+\varepsilon}{\rho_k} + \frac{\|x_{k-1}\|}{\rho_k}, \quad \forall x \in \bar{\Omega}_\varepsilon.$$

By the conditions $\frac{1}{\rho_k} \to 0$ and $\frac{\|x_{k-1}\|}{\rho_k} \to 0$ when k tends to infinity, it follows that there exists a positive integer k_0 such that for every $k \ge k_0$, we have

$$\frac{M+\varepsilon}{\rho_k} \le \frac{\delta}{3}, \quad \text{and} \quad \frac{\|x_{k-1}\|}{\rho_k} \le \frac{\delta}{3}.$$

Hence, for all $k \ge k_0$, we have

$$\left\|\tilde{F}_{k,K}^{\text{nat}}(x) - F_K^{\text{nat}}(x)\right\| \le \frac{2}{3}\delta, \quad \forall x \in \bar{\Omega}_\varepsilon$$

which implies

$$\sup_{\bar{\Omega}_s}\left\|\tilde{F}_{k,K}^{\text{nat}}(x) - F_K^{\text{nat}}(x)\right\| \le \frac{2}{3}\delta < \delta.$$

Consequently, by applying the condition (8)

$$\left(\tilde{F}_{k,K}^{\text{nat}}\right)^{-1}(0) \subseteq \left(F_K^{\text{nat}}\right)^{-1}(0) + \varepsilon B(1), \quad \forall k \ge k_0.$$

Moreover, we can easily check that $\text{SOL}\left(K, F^{(k)}\right)$ coincide with $\text{SOL}\left(K, \tilde{F}^{(k)}\right)$, therefore, this fact implies

$$\{x_k\} = \left(\tilde{F}_{k,K}^{\text{nat}}\right)^{-1}(0) \subseteq \left(F_K^{\text{nat}}\right)^{-1}(0) + \varepsilon B(1), \ \forall k \ge k_0. \tag{9}$$

Hence, for any $k \ge k_0$, it holds that

$$\operatorname{dist}\left(x_k \,\middle|\, \left(F_K^{\text{nat}}\right)^{-1}(0)\right) = \inf\left\{\|x_k - y\| : y \in \left(F_K^{\text{nat}}\right)^{-1}(0)\right\} \le \varepsilon.$$

This implies that

$$\lim_{k\to+\infty} \operatorname{dist}\left(x_k \,|\, \mathrm{S}\right) = \lim_{k\to+\infty} \operatorname{dist}\left(x_k \,\middle|\, \left(F_K^{\text{nat}}\right)^{-1}(0)\right) = 0.$$

Since the set $\left\{x_1, \ldots, x_{k_0}\right\}$ is finite, it is also bounded. From (9), since $\mathrm{S} = \left(F_K^{\text{nat}}\right)^{-1}(0)$ is contained in the ball with radius M, we have that

$$\|x_k\| \le M + \varepsilon, \forall k \ge k_0.$$

Together, we obtain the boundedness of $\{x_k\}$.

Finally, let x^* be any accumulation point of $\{x_k\}$, then there is a subsequence $\left\{x_{k_i}\right\}$ of $\{x_k\}$ converging to x^*. For any fixed i, x_{k_i} is a solution to $\mathrm{VI}\left(K, F^{(k_i)}\right)$, thus satisfies the following inequality for all x in K

$$\left\langle F^{(k_i)}\left(x_{k_i}\right), x - x_{k_i}\right\rangle \ge 0,$$

or, equivalently,

$$\left\langle F\left(x_{k_i}\right) + \frac{x_{k_i} - x_{k_i - 1}}{\rho_{k_i}}, x - x_{k_i}\right\rangle \ge 0.$$

In the preceding inequality, since $\{x_k\}$ is bounded and $\rho_{k_i} \to \infty$ as $i \to \infty$, we obtain the following inequality for all x in K when $i \to \infty$

$$\left\langle F\left(x^*\right), x - x^*\right\rangle \ge 0,$$

This shows that x^* is a solution to the original problem.

Remark 3.1. We can construct a sequence $\{\rho_k\}$ that satisfies the conditions (6) in the following way: First, we choose a positive number ρ_1 and an arbitrary vector x_0 in $\mathbb{R}^n$. We will then obtain a unique solution x_1 to the $\mathrm{VI}\left(K, F^{(1)}\right)$ where

$$F^{(1)}(x) = \rho_1 F(x) + x - x_0.$$

Next, we choose a positive number ρ_2 satisfying

$$\rho_2 \ge \max\left\{\rho_1, 2\right\} \quad \text{and} \quad \frac{\|x_1\|}{\rho_2} \le \frac{1}{2},$$

and we continue to obtain the unique solution x_2 to the VI $\left(K, F^{(2)}\right)$ where

$$F^{(2)}(x) = \rho_2 F(x) + x - x_1.$$

Continuing to choose a positive number ρ_3 satisfying

$$\rho_3 \geq \max\{\rho_2, 3\} \quad \text{and} \quad \frac{\|x_2\|}{\rho_3} \leq \frac{1}{3},$$

We obtain the unique solution x_3 to $\text{VI}\left(K, F^{(3)}\right)$ where

$$F^{(3)}(x) = \rho_3 F(x) + x - x_2.$$

By doing this process consecutively, we will construct a sequence $\{\rho_k\}$ which is increasing and satisfies

$$\rho_k \geq k, \quad \frac{\|x_{k-1}\|}{\rho_k} \leq \frac{1}{k}, \quad \forall k \geq 2.$$

Therefore, the sequence $\{\rho_k\}$ satisfies the conditions (6).

The following example shows that the boundedness condition of $\text{SOL}(K, F)$ in Theorem 3.2 cannot be dropped.

Example 1. Let $K = \mathbb{R}^2_+$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x) = Mx + q, \quad x \in \mathbb{R}^2$$

where

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Obviously, F given above is a P_0-function on $\mathbb{R}^2$ so will be a P_0-function on $\mathbb{R}^2_+$ in the classical sense. Then, the $\text{VI}(K, F)$ becomes the complementarity problem, that is to find $\bar{x} = \left(\bar{x}_1, \bar{x}_2\right)$ in $\mathbb{R}^2_+$ satisfying

$$M\bar{x} + q \geq 0 \quad \text{and} \quad \langle \bar{x}, M\bar{x} + q \rangle = 0.$$

More precisely, $\bar{x}$ must satisfy

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \bar{x}_2 - 1 \\ 0 \end{pmatrix} \geq 0, \quad \text{and} \quad \bar{x}_1\left(\bar{x}_2 - 1\right) = 0.$$

From this point, we have that

$$\text{SOL}(K, F) = \left\{\left(x_1, 1\right) : x_1 \geq 0\right\} \cup \left\{\left(0, x_2\right) : x_2 \geq 1\right\}.$$

We see that this set is nonempty and unbounded. Next, we will construct the sequence of solutions $\{x_k\}$ to the perturbed problems $\text{VI}\left(K, F^{(k)}\right)$ by using the proximal point algorithm and examine its iteration. The iteration can be established generally in the following way: given an arbitrarily positive number ρ_1 and an initial point $x_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ such that x_2^0 belongs to (0,1) and x_1^0 satisfies $x_1^0 + \rho_1 \le 0$, for each positive integer $k \ge 2$, we choose ρ_k as a number satisfying

$$\rho_k \ge \max\{\rho_{k-1}, k\},$$
$$\frac{\|x_{k-1}\|}{\rho_k} \le \frac{1}{2^{k-1}},$$

where x_{k-1} is the unique solution to the $\mathrm{VI}\left(K, F^{(k-1)}\right)$. By induction, we obtain the iteration $\{x_k\}$ satisfying all the following conditions:

$$x_2^k = x_2^0,$$
$$x_1^k = x_1^{k-1} + \rho_k\left(1 - x_2^0\right),$$
$$x_1^k > 0,$$

for any k, where $x_k = \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix}$. The construction of $\{\rho_k\}$ and $\{x_k\}$ give us the following properties:

$$\lim_{k\to+\infty} \rho_k = +\infty \text{ and } \lim_{k\to+\infty} \frac{x_{k-1}}{\rho_k} = 0.$$

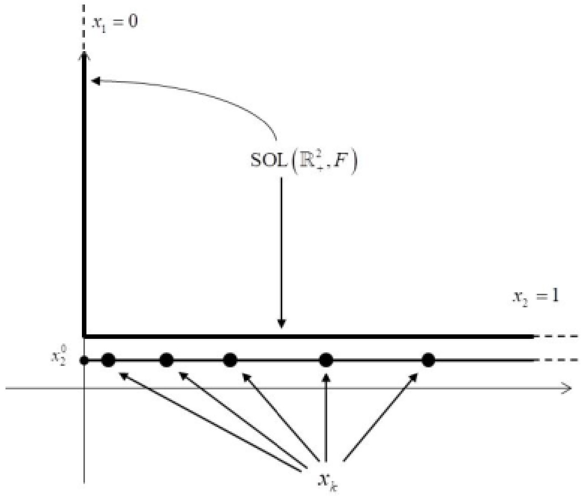
In other words, the iteration $\{x_k\}$ generated by the proximal point algorithm in this example satisfied all the conditions of Theorem 3.2. Moreover, this iteration also satisfies $x_2^k = x_2^0$ for all $k \in \mathbb{N}$ hence it lies on the ray $\left\{\left(x_1, x_2^0\right) : x_1 \ge 0\right\}$. This leads to

$$\mathrm{dist}\left(x_k \mid \mathrm{S}\right) = 1 - x_2^0, \quad \forall k \in \mathbb{N}$$

And obviously, this implies

$$\lim_{k\to+\infty} \mathrm{dist}\left(x_k \mid \mathrm{S}\right) = 1 - x_2^0 \ne 0.$$

We can illustrate this easily through the following figure.



We next consider two particular cases. First, if the set K is bounded, we will obtain the nonemptiness and boundedness of $\mathrm{SOL}(K,F)$. Moreover, we can drop the assumption that $\left\{\frac{x_{k-1}}{\rho_k}\right\}$ converges to 0 as k tends to infinity. In summary, we have the following corollary.

Corollary 3.1. Let $\mathrm{VI}(K,F)$ be a generalized P_0 problem where F is continuous on $\mathbb{R}^n$. Assume further that the set K is bounded. Then, if the sequence $\{\rho_k\}$ satisfies $\rho_k \to +\infty$ as $k \to +\infty$, it holds that

$$\lim_{k\to+\infty} \mathrm{dist}\left(x_k \mid \mathrm{S}\right) = 0.$$

If $\mathrm{SOL}(K,F)$ is a singleton, the convergence of the iteration is then obtained.

Corollary 3.2. Let $\mathrm{VI}(K,F)$ be a generalized P_0 problem where F is continuous on $\mathbb{R}^n$. Assume further that the $\mathrm{VI}(K,F)$ has a unique solution x^*. Then, if the sequence $\{\rho_k\}$ satisfies the conditions stated in Theorem 3.2, it holds that

$$\lim_{k\to+\infty} x_k = x^*.$$

Remark 3.2. Proposition 3.5.10 (a) in Facchinei and Pang (2003) gives us the uniqueness of a solution to the generalized P problems, then we can apply the PPA to the generalized P problems and obtain similar results.

4. Conclusion

We have applied the proximal point algorithm to the generalized P_0 problem and obtained several properties for the iteration generated by the algorithm, including the convergence result. Furthermore, we have constructed an example to illustrate the conditions stated in the main theorem.Open problems remain in this topic. For instance, it is of interest to study the following questions:

(Q1) Is the assumption on the convergence to 0 of the sequence $\left\{\frac{\|x_{k-1}\|}{\rho_k}\right\}$ in Theorem 3.2 a redundant one?

(Q2) Are there any other conditions for the sequence $\{\rho_k\}$ under which we obtain the property for the $\{x_k\}$ stated in Theorem 3.2 and obtain the convergence of $\{x_k\}$?

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

❖ ***Acknowledgement:*** *This research is funded by Ho Chi Minh City University of Education Foundation for Science and Technology under grant number CS.2021.19.01TD.*

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THUẬT TOÁN ĐIỂM GẦN KỀ CHO LỚP BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN P_0 SUY RỘNG

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Ngày nhận bài: 20-02-2023; ngày nhận bài sửa: 23-3-2023; ngày duyệt đăng: 24-03-2023

TÓM TẮT

Bài báo này nghiên cứu thuật toán điểm gần kề cho lớp bài toán bất đẳng thức biến phân P_0 suy rộng. Bằng cách sử dụng kết quả về tính nửa liên tục trên của các toán tử đơn diệp yếu chúng tối chứng minh dãy lặp sinh bởi thuật toán là bị chặn và bám vào tập nghiệm của bài toán ban đầu và mỗi điểm tụ của dãy lặp là nghiệm của bài toán đã cho dưới giả thiết tập nghiệm bị chặn. Chúng tôi cũng đưa ra một ví dụ chỉ ra sự cần thiết của tính bị chặn của tập nghiệm.

Từ khóa: hội tụ; ánh xạ tự nhiên; t hàm P_0; hàm *P*; thuật toán điểm gần kề; toán tử đơn điệp; bất đẳng thức biến phân