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Research Article

REGULARITY RESULTS FOR *(p,q)*-LAPLACE TYPE EQUATIONS IN GENERALIZED MORREY SPACES

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ABSTRACT

The quasi-linear non-uniformly elliptic problems were motivated by minimizing problems for non-standard integral energy functionals, which can be applied to many applications in sciences such as fluid dynamics, nonlinear elasticity, and physics. A typical example of this type of problem may be seen as the (p,q)-Laplace equation. In this paper, we establish some gradient estimates via fractional maximal operators for a class of (p,q)-Laplace type equations in generalized Morrey spaces. The global regularity results were obtained in two steps. In the first step, we construcedt the gradient estimate in the setting of weighted Lorentz spaces. The regularity result in Morrey spaces were obtained in the second step.

Keywords: generalized Morrey spaces; Non-uniformly elliptic problems; *(p,q)*-Laplace equation; regularity; weighted Lorentz spaces

1. Introduction

The main goal of this paper is to establish some global gradient estimates for non-uniformly elliptic problems which have a typical version as below:

$$-\operatorname{div}\big(V(x,\nabla u)\big) = -\operatorname{div}\big(V(x,F)\big) \quad \text{in } \Omega, \tag{1.1}$$

where the vector field V is defined by

$$V(x,\xi) = p|\xi|^{p-1}\xi + qa(x)|\xi|^{q-1}\xi, \quad \xi \in \mathbb{R}^n. \tag{1.2}$$

Here, the domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is open bounded, and the given data $F:\Omega \to \mathbb{R}^n$ is a vector field. The coefficient function $a \in C^{0,\sigma}(\Omega,\mathbb{R}^+)$ for some $\sigma \in (0,1]$ and two parameters p, q satisfy the following assumption:

$$1 < p < q \leq \left(1 + \frac{\sigma}{n}\right)p. \tag{1.3}$$

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The *(p,q)*-Laplace equation in (1.1) can be seen as the Euler-Lagrange equation of the following functional:

$$w \mapsto E(w) := \int_{\Omega} \Phi(x, \nabla w)dx - \int_{\Omega} \langle V(x,F), \nabla w \rangle dx \ ,$$

where the function Φ is defined by the Euclidean norm of V in (1.2) as follows:

$$\Phi(x,\xi) := \left|\xi\right|^p + a(x)\left|\xi\right|^q, \ (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{1.4}$$

The mapping E is called double phase functional which was first studied by Zhikov (1986, 1995) to describe the change of ellipticity according to the range of functions a. Related to the regularity of solutions to (1.1), Marcellini (1991), Esposito, Leonetti, & Mingione (2004) and(Colombo, & Mingione (2015, 2016) studied the local gradient estimates as follows:

$$\Phi(x,F) \in L^{\gamma}_{\mathrm{loc}} \Rightarrow \Phi(x,\nabla u) \in L^{\gamma}_{\mathrm{loc}}, \quad \text{for all}\ \gamma > 1. \tag{1.5}$$

After that, there have been many studies on the regularity for non-uniformly elliptic problems, such as Byun and Oh (2017), Baroni, Colombo, & Mingione (2018), De Filippis and Mingione (2020), Beck and Mingione (2020), Byun and Lee (2021), Tran and Nguyen (2021), Tran (2022), Dang and Pham (2022), and Nguyen et al. (2023).

It is worth mentioning that in a special case $a \equiv 0$, equation (1.1) is well known as *p*-Laplace equation which has attracted the interest of many researchers in recent years. In particular, we are interested in the global Calderón-Zygmund estimates of the solutions in various generalized Lebesgue spaces (Nguyen & Tran, 2020; Tran, Nguyen, & Nguyen, 2021; Tran & Nguyen (2020, 2023). In these papers, the key technique comes from the construction of level-set inequalities by combining comparison estimates and Vitali covering lemma. This technique was introduced by Tran and Nguyen (2021), Tran and Nguyen (2019, 2022), and Nguyen, Tran, and Huynh (2021). Motivated by these works, we continue to investigate the regularity results for *(p,q)*-Laplace type equations in some more general function spaces.

In the present paper, we study a class of *(p,q)*-Laplace type equations which are more general than (1.1). Roughly speaking, the vector-valued function V in (1.2) will be generalized by two nonlinear operators $\mathcal{A}, \mathcal{B}$. In other words, we consider the following equations:

$$\begin{cases} -\mathrm{div}\left(\mathcal{A}(x,\nabla u)\right) &= -\mathrm{div}\left(\mathcal{B}(x,F)\right), & \text{in } \Omega, \\ \qquad u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

Where the nonlinear operators $\mathcal{A}, \mathcal{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following conditions:

$$\begin{cases} \left(\left|\mathcal{A}(x,\xi)\right| + \left|\mathcal{B}(x,\xi)\right| + \left|\partial_{\xi}\mathcal{A}(x,\xi)\right|\left|\xi\right|\right)\left|\xi\right| \le L\Phi(x,\xi), \\ \varkappa\left(\left|\xi\right|^{p-2} + a(x)\left|\xi\right|^{q-2}\right)\left|z\right|^2 \le \langle \partial_{\xi}\mathcal{A}(x,\xi)z, z\rangle, \\ \left|\mathcal{A}(x_1,\xi) - \mathcal{A}(x_2,\xi)\right|\left|\xi\right| \le L\left|a(x_1) - a(x_2)\right|\left|\xi\right|^q, \end{cases} \tag{1.7}$$

for all $x,\ x_1,\ x_2 \in \Omega$ and $\xi,\ z \in \mathbb{R}^n \setminus \{0\}$, with two given constants $\varkappa, L > 0$. More precisely, we establish the following regularity result:

$$\mathbf{M}_{\beta}(\Phi(x,\mathbf{F})) \in \mathcal{M}^{s,\Psi}(\Omega) \Rightarrow \mathbf{M}_{\beta}\left(\Phi(x,\nabla u)\right) \in \mathcal{M}^{s,\Psi}(\Omega), \tag{1.8}$$

where $\mathcal{M}^{s,\Psi}(\Omega)$ denotes the generalized Morrey space and $\mathbf{M}_{\beta}$ is the fractional maximal operators that will be defined later. To prove (1.8), we first apply the distribution function via fractional maximal operators for weak solutions to (1.6). An interesting point is that the quasi-norm in generalized weighted Lorentz spaces can be presented by the distribution function mentioned above. Therefore we may obtain the gradient estimate in generalized weighted Lorentz spaces. Then, we choose a very special weight and use the dyadic decomposition of $\mathbb{R}^n$ to establish the result in Morrey spaces. We refer to Nguyen *et al.*, (2023) or Tran *et al.* (2022) for a similar method applying to some classes of steady Stokes systems.

The rest of the paper will be organized as follows. In the next section, we recall some notations and definitions related to weak solutions, fractional maximal operators, distribution functions, and several functional spaces. In the last section, we prove the gradient estimates in weighted Lorentz spaces and generalized Morrey spaces.

2. Preliminaries

Let us first introduce some notations and definitions that are considered throughout the paper. The diameter of the open bounded domain $\Omega \subset \mathbb{R}^n$ will be denoted by $\operatorname{diam}(\Omega)$. We denote an open ball in $\mathbb{R}^n$ with center x_0 and radius $R > 0$ by

$$B_R(x_0) := \left\{\xi \in \mathbb{R}^n : \left|\xi - x_0\right| < R\right\}.$$

The union of Ω and a ball will denoted by $\Omega_R(x_0) = \Omega \cap B_R(x_0)$. Moreover, we use notation $\{|h| > \tau\}$ instead of $\{x \in \Omega : |h(x)| > \tau\}$. On the other hand, we write $\mathcal{L}^n(A)$ for the Lebesgue measure of a set $A \subset \mathbb{R}^n$. With the coefficient function $a \in C^{0,\sigma}(\Omega, \mathbb{R}^+)$, we write

$$[a]_{\sigma;\Omega} = \sup_{x,y\in\Omega;\, x\neq y} \frac{\left|a(x) - a(y)\right|}{\left|x - y\right|^{\sigma}}.$$

The existence of solutions to (1.6) is studied in Musielak-Orlicz-Sobolev spaces, see Benkirane and El Vally (2014). Let us now recall the definition of Musielak-Orlicz-Sobolev spaces according to the operator Φ in (1.4). We notice here that we still denote by Φ for the following function:

$$\Phi(x,\xi) := \left|\xi\right|^p + a(x)\left|\xi\right|^q, \ (x,\xi) \in \Omega \times \mathbb{R}.$$

Definition 2.1. (Musielak-Orlicz-Sobolev spaces) Let $h : \Omega \to \mathbb{R}$ be a measurable function, we say h belongs to the Musielak-Orlicz class $O^{\Phi}(\Omega)$ if it satisfies

$$\int_{\Omega} \Phi(x,h)dx < +\infty.$$

The Musielak-Orlicz space $L^{\Phi}(\Omega)$ is the smallest vectorial space containing $O^{\Phi}(\Omega)$ equipped to the norm $\|.\|_{L^{\Phi}(\Omega)}$ as below:

$$\|h\|_{L^{\Phi}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(x, \frac{|h(x)|}{\lambda}\right)dx \le 1\right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is the set of all measurable functions $h \in L^{\Phi}(\Omega)$ such that $|\nabla h| \in L^{\Phi}(\Omega)$. The norm of the space $W^{1,\Phi}(\Omega)$ is given by

$$\|h\|_{W^{1,\Phi}(\Omega)} = \|h\|_{L^{\Phi}(\Omega)} + \|\nabla h\|_{L^{\Phi}(\Omega)}.$$

Furthermore, we define by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$.

Definition 2.2. (Weak solution) A function $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution to (1.6) under assumptions (1.3) and (1.7) if it satisfies

$$\int_{\Omega} \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle dx = \int_{\Omega} \langle \mathcal{B}(x,F), \nabla\varphi\rangle dx$$

for every test function $\varphi \in C_c^{\infty}(\Omega)$, where notation $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^n$.

The existence and uniqueness of weak solutions to (1.6) is well known in Marcellini (1991) and Colombo and Mingione (2016). Moreover, the authors prove that we can test the variational formula $\varphi \in W_0^{1,\Phi}(\Omega)$ instead of $\varphi \in C_c^{\infty}(\Omega)$.

Lemma 2.3. *(Colombo & Mingione, 2016) Let $u \in W_0^{1,\Phi}(\Omega)$ be a weak solution to (1.6) under conditions (1.3) and (1.7) with given data $F \in W^{1,\Phi}(\Omega)$. Then the following formula*

$$\int_{\Omega} \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle dx = \int_{\Omega} \langle \mathcal{B}(x,F), \nabla\varphi\rangle dx$$

holds for every test function $\varphi \in W_0^{1,\Phi}(\Omega)$.

Definition 2.4. (Fractional maximal operator) We define $\mathbf{M}_{\beta}$ the maximal operator with order $0 \le \beta \le n$, which is the operator given by

$$\mathbf{M}_{\beta}h(x) = \sup_{r>0} \frac{r^{\beta}}{\mathcal{L}^n\left(B_r(x)\right)} \int_{B_r(x)} \left|h(\xi)\right| d\xi, \quad \forall x \in \mathbb{R}^n,$$

for all $h \in L^1_{\text{loc}}\left(\mathbb{R}^n\right)$.

It is well known that when $\beta = 0$, then $\mathbf{M}_{\beta} = \mathbf{M}_0$ is exactly the Hardy-Littlewood operator $\mathbf{M}$ defined by

$$\mathbf{M}h(x) = \sup_{r>0} \frac{1}{\mathcal{L}^n\left(B_r(x)\right)} \int_{B_r(x)} \left|h(\xi)\right| d\xi, \ \forall x \in \mathbb{R}^n,$$

A nice feature of a fractional maximal operator $\mathbf{M}_{\beta}$ is its bounded property.

Lemma 2.5. *(Tran & Nguyen, 2021, Lemma 2.8) For every $0 \le \beta < n$ and $1 \le s < \frac{n}{\beta}$, there exists a constant $C > 0$ such that*

$$\mathcal{L}^n\left(\left\{x \in \mathbb{R}^n : \mathbf{M}_{\beta}h(x) > \lambda\right\}\right) \le \left(\frac{C}{\lambda^s} \int_{\mathbb{R}^n} \left|h(x)\right|^s dx\right)^{\frac{n}{n-s\beta}},$$

for all $h \in L^s\left(\mathbb{R}^n\right)$ and for all $\lambda > 0$.

Definition 2.6. (Muckenhoupt weights) Let $1 \le v \le \infty$, we say that a weight $\omega \in L^1_{\text{loc}}\left(\mathbb{R}^n; \mathbb{R}^+\right)$ belongs to the class of Muckenhoupt weights $\mathbf{A}_v$ if

$$[\omega]_{\mathbf{A}_v} := \sup_{B_r(x) \subset \mathbb{R}^n} \left(r^{-n} \int_{B_r(x)} \omega(\xi) d\xi\right) \left(r^{-n} \int_{B_r(x)} \omega(\xi)^{-\frac{1}{v-1}} d\xi\right)^{v-1} < \infty,$$

when $1 < v < \infty$,

$$[\omega]_{\mathbf{A}_1} := \sup_{B_r(x) \subset \mathbb{R}^n} \left[\left(r^{-n} \int_{B_r(x)} \omega(\xi) d\xi\right) \sup_{\xi \in B_r(x)} \frac{1}{\omega(\xi)}\right] < \infty,$$

when $v = 1$ and there exists positive constants C_1, C_2, and τ_1, τ_2 satisfying

$$C_1 \left(\frac{\mathcal{L}^n(E)}{\mathcal{L}^n(B)}\right)^{\tau_1} \le \frac{\omega(E)}{\omega(B)} \le C_2 \left(\frac{\mathcal{L}^n(E)}{\mathcal{L}^n(B)}\right)^{\tau_2},$$

when $v = \infty$, for any ball B in $\mathbb{R}^n$, and all measurable subsets E of B. Here, $\omega(E)$ is defined by

$$\omega(E) = \int_E \omega(x) dx.$$

In this case, we denote $[\omega]_{\mathbf{A}_\infty} = (C_1, C_2, \tau_1, \tau_2)$.

Definition 2.7. (Distribution functions) Let $\omega \in \mathbf{A}_\infty$ and h be a locally integrable function on $\mathbb{R}^n$. A weighted distribution function $\mathbf{d}_h^\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\mathbf{d}_h^\omega(\lambda) := \omega\left(\left\{x \in \Omega : \left|h(x)\right| > \lambda\right\}\right), \quad \lambda \geq 0.$$

Moreover, we consider a new distribution function that corresponds to the fractional maximal operators. More precisely, we define

$$\mathbf{D}_h^{\beta,\omega}(\lambda) := \mathbf{d}_{\mathbf{M}_\beta h}^\omega(\lambda) = \omega\left(\left\{x \in \Omega : \left|\mathbf{M}_\beta h(x)\right| > \lambda\right\}\right), \quad \lambda \geq 0. \tag{2.1}$$

Next, to define the generalized weighted Lorentz space, we consider a new weight $\varsigma \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ a non-decreasing function $\mathcal{K}$ defined by

$$\mathcal{K}(\lambda) = \int_0^\lambda \varsigma(\lambda) d\lambda, \quad \lambda \geq 0. \tag{2.2}$$

The weighted Lorentz spaces can be defined as below, see Carro, Raposo, and Soria (2007).

Definition 2.8. (Generalized weighted Lorentz spaces) Let $\omega \in \mathbf{A}_\infty$ and $\mathcal{K}$ defined by (2.2). Let $s \in (0,\infty)$ and $0 < t \leq \infty$, the generalized weighted Lorentz space $\Lambda^{s,t}_{\varsigma,\omega}(\Omega)$ is the set of all functions h such that

$$\left\|h\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} := \left[s\int_0^\infty \lambda^{t-1}\left[\mathcal{K}\left(\mathbf{d}_h^\omega(\lambda)\right)\right]^{\frac{t}{s}} d\lambda\right]^{\frac{1}{t}}$$

if $t < \infty$ and

$$\left\|h\right\|_{\Lambda^{s,\infty}_{\varsigma,\omega}} := \sup_{\lambda>0} \lambda\left[\mathcal{K}\left(\mathbf{d}_h^\omega(\lambda)\right)\right]^{\frac{1}{s}}$$

if $t = \infty$.

We remark here that $\left\|\cdot\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)}$ is a quasi-norm in the generalized weighted Lorentz space $\Lambda^{s,t}_{\varsigma,\omega}(\Omega)$ if and only if there exists $\beta_2 > 0$ such that

$$\mathcal{K}(2\lambda) \leq \beta_2 \mathcal{K}(\lambda), \quad \forall \lambda \geq 0.$$

On the other hand, if $\varsigma \equiv 1$ and $\omega \equiv 1$ then the generalized weighted Lorentz space $\Lambda^{s,t}_{\varsigma,\omega}(\Omega)$ becomes the classical Lorentz space $L^{s,t}(\Omega)$.

Definition 2.9. ($\Psi -$generalized Morrey spaces) Let $\Psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. Then, a measurable mapping $h : \Omega \to \mathbb{R}$ is said to belong to the $\Psi -$generalized Morrey space $\mathcal{M}^{s,\Psi}(\Omega)$ with $s \in (0,\infty)$ if

$$\left\|h\right\|_{\mathcal{M}^{s,\Psi}(\Omega)} := \sup_{x\in\Omega;\ 0<\varrho<\mathrm{diam}(\Omega)} \left(\frac{1}{\Psi(x,\varrho)}\int_{\Omega_\varrho(x)} \left|h(\xi)\right|^s d\xi\right)^{\frac{1}{s}} < \infty. \quad (2.3)$$

For simplicity of notation, we use $data$ to stand for the set of parameters that will affect the constant dependence in our statements below. In the sequel, we use

$$data \equiv data\left(n,q,p,\sigma,\varkappa,L,\beta,[a]_{\sigma;\Omega},\|a\|_{L^\infty(\Omega)},\|\Phi(x,\nabla u)\|_{L^1(\Omega)},[\omega]_{\mathbf{A}_\infty}\right).$$

Finally, we always denote C a general positive constant that depends on $data$.

3. Results

The gradient estimate in generalized Morrey spaces was obtained in two steps. The first one is the estimate in weighted Lorentz spaces. The key point of this work was based on the weighted level-set inequality of data and the gradient of weak solutions. In particular, thanks to Theorem 4.3 by Tran and Nguyen (2021), for every $\kappa > 0$ it is possible to find $\varepsilon_0 = \varepsilon_0(\kappa) \in (0,1)$, $\eta = \eta(\kappa) > 0$ and a positive constant C such that the following inequality:

$$\omega\left(\left\{\mathbf{M}_\beta(\Phi(x,\nabla u)) > \varepsilon^{-\kappa}\lambda, \mathbf{M}_\beta(\Phi(x,F)) \le \varepsilon^{\eta}\lambda\right\} \cap \Omega\right)$$
$$\le C^*\varepsilon\omega\left(\left\{\mathbf{M}_\beta(\Phi(x,\nabla u)) > \lambda\right\} \cap \Omega\right) \quad (3.1)$$

holds for every $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$. This level-set inequality is also called good-λ inequality. For more information, read Tran (2022) and Dang and Pham (2022) for the proofs of (3.1). We remark that this level-set inequality can be proved under an additional assumption on the boundary of the domain. More precisely, we assume that $\partial\Omega$ is $C^{1,\sigma}$.

Theorem 3.1. *Let Ω be an open bounded domain in $\mathbb{R}^n$ such that $\partial\Omega$ is $C^{1,\sigma}$. Assume that $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution to (1.6) under assumptions (1.3) and (1.7) with given data $F \in W^{1,\Phi}(\Omega)$. Suppose moreover that $\omega \in \mathbf{A}_\infty$, $\varsigma \in L^1_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R}^+)$ and $\mathcal{K}$ is given as in (2.1) satisfying*

$$\theta_1\mathcal{K}(\lambda) \le \mathcal{K}(2\lambda) \le \theta_2\mathcal{K}(\lambda), \quad \forall\lambda \ge 0, \quad (3.2)$$

for some $\theta_2 > \theta_1 > 1$. Then for every $\beta \in [0,n)$, $s \in (0,\infty)$ and $0 < t \le \infty$, the following gradient estimate

$$\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} \le C\left\|\mathbf{M}_\beta(\Phi(x,F))\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} \quad (3.3)$$

holds true with a constant $C = C(data,s,t,\theta_1,\theta_2)$.

Proof. Firstly, for every $\kappa > 0$ one can find some contants $\varepsilon_0 \in (0,1)$ and $\eta > 0$ such that (3.1) holds for every $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$. It leads to

$$\omega\left(\left\{\mathbf{M}_{\beta}(\Phi(x,\nabla u)) > \varepsilon^{-\kappa}\lambda\right\} \cap \Omega\right) \leq \omega\left(\left\{\mathbf{M}_{\beta}(\Phi(x,F)) > \varepsilon^{\eta}\lambda\right\} \cap \Omega\right) + C^{*}\varepsilon\omega\left(\left\{\mathbf{M}_{\beta}(\Phi(x,\nabla u)) > \lambda\right\} \cap \Omega\right),$$

which can be rewritten in the form of distribution functions in (2.1) as below:

$$\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\varepsilon^{-\kappa}\lambda) \leq C^{*}\varepsilon\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda) + C^{*}\mathbf{D}^{\beta,\omega}_{\Phi(x,F)}(\varepsilon^{\eta}\lambda), \tag{3.4}$$

for every $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$. Thanks to assumption (3.2), we show that

$$\mathcal{K}(\lambda_1 + \lambda_2) \leq \theta_2\left(\mathcal{K}(\lambda_1) + \mathcal{K}(\lambda_2)\right), \quad \forall \lambda_1, \lambda_2 \geq 0. \tag{3.5}$$

Indeed, we can suppose that $0 < \lambda_1 < \lambda_2$, since $\mathcal{K}$ is non-decreasing, there holds

$$\mathcal{K}(\lambda_1 + \lambda_2) \leq \mathcal{K}(2\lambda_2) \leq \theta_2\mathcal{K}(\lambda_2) \leq \theta_2\left(\mathcal{K}(\lambda_1) + \mathcal{K}(\lambda_2)\right).$$

From the estimates in (3.4) and (3.5), one obtains that

$$\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\varepsilon^{-\kappa}\lambda)\right) \leq \theta_2\left(\mathcal{K}\left(C^{*}\varepsilon\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda)\right) + \mathcal{K}\left(C^{*}\mathbf{D}^{\beta,\omega}_{\Phi(x,F)}(\varepsilon^{\eta}\lambda)\right)\right). \tag{3.6}$$

On the other hand, we can fix $m \in \mathbb{N}$ such that $2^{m-1} < C^{*} \leq 2^{m}$. By (3.2) for every $\tau > 0$, we deduce that

$$\mathcal{K}(C^{*}\tau) \leq \mathcal{K}\left(2^{m}\tau\right) \leq \theta_2^{m}\mathcal{K}(\tau),$$

together with (3.6) one has

$$\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\varepsilon^{-\kappa}\lambda)\right) \leq \theta_2^{m+1}\left(\mathcal{K}\left(\varepsilon\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda)\right) + \mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,F)}(\varepsilon^{\eta}\lambda)\right)\right). \tag{3.7}$$

Let $0 < t < \infty$ and $0 < s < \infty$, we now consider the quasi-norm in the weighted Lorentz space $\Lambda^{s,t}_{\varsigma,\omega}(\Omega)$. We can present as follows:

$$\begin{aligned}\left\|\mathbf{M}_{\beta}(\Phi(x,\nabla u))\right\|^{t}_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} &= s\int_0^{\infty}\lambda^{t-1}\left[\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda)\right)\right]^{\frac{t}{s}}d\lambda \\ &= \varepsilon^{-t\kappa}s\int_0^{\infty}\lambda^{t-1}\left[\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\varepsilon^{-\kappa}\lambda)\right)\right]^{\frac{t}{s}}d\lambda.\end{aligned}$$

Substituting (3.7) into this formula, we obtain that

$$\begin{aligned}\left\|\mathbf{M}_{\beta}(\Phi(x,\nabla u))\right\|^{t}_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} &\leq C\varepsilon^{-t\kappa}s\int_0^{\infty}\lambda^{t-1}\left[\mathcal{K}\left(\varepsilon\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda)\right)\right]^{\frac{t}{s}}d\lambda \\ &\quad + C\varepsilon^{-t\kappa}s\int_0^{\infty}\lambda^{t-1}\left[\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,F)}(\varepsilon^{\eta}\lambda)\right)\right]^{\frac{t}{s}}d\lambda.\end{aligned} \tag{3.8}$$

We may now define $k \in \mathbb{N}$ such that

$$\frac{1}{2} < 2^{k}\varepsilon \leq 1 \Leftrightarrow \log_2\left(\frac{1}{\varepsilon}\right) - 1 < k \leq \log_2\left(\frac{1}{\varepsilon}\right).$$

Then for every $\tau > 0$, from the assumption (3.2), we can assert that

$$\mathcal{K}(\varepsilon\tau) \leq \frac{1}{\theta_1^k}\mathcal{K}\left(2^k\varepsilon\tau\right) \leq \frac{1}{\theta_1^k}\mathcal{K}(\tau) \leq \frac{1}{\theta_1^{\log_2(\frac{1}{\varepsilon})-1}}\mathcal{K}(\tau),$$

which from (3.8) allows us to imply that

$$\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|^t_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} \leq C\varepsilon^{-t\kappa}\left(\frac{1}{\theta_1^{\log_2\left(\frac{1}{\varepsilon}\right)-1}}\right)^{\frac{t}{s}} s\int_0^\infty \lambda^{t-1}\left[\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,\nabla u)}(\lambda)\right)\right]^{\frac{t}{s}}d\lambda$$
$$+ C\varepsilon^{-t\kappa}\varepsilon^{-t\eta}s\int_0^\infty \lambda^{t-1}\left[\mathcal{K}\left(\mathbf{D}^{\beta,\omega}_{\Phi(x,F)}(\lambda)\right)\right]^{\frac{t}{s}}d\lambda.$$

This inequality can be rewritten in the form of a quasi-norm in weighted Lorentz space $\Lambda^{s,t}_{\varsigma,\omega}(\Omega)$ as follows:

$$\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|^t_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} \leq C\varepsilon^{-t\kappa}\left(\frac{1}{\theta_1^{\log_2\left(\frac{1}{\varepsilon}\right)-1}}\right)^{\frac{t}{s}}\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|^t_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)}$$
$$+ C\varepsilon^{-t(\kappa+\eta)}\left\|\mathbf{M}_\beta(\Phi(x,F))\right\|^t_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)}.$$

Using a fundamental inequality, the above estimate implies to

$$\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)} \leq C\varepsilon^{-\kappa}\left(\frac{1}{\theta_1^{\log_2\left(\frac{1}{\varepsilon}\right)-1}}\right)^{\frac{1}{s}}\left\|\mathbf{M}_\beta(\Phi(x,\nabla u))\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)}$$
$$+ C\varepsilon^{-(\kappa+\eta)}\left\|\mathbf{M}_\beta(\Phi(x,F))\right\|_{\Lambda^{s,t}_{\varsigma,\omega}(\Omega)}. \tag{3.9}$$

We now apply this argument again to show that (3.9) still holds for $s \in (0,\infty)$ and $t = \infty$. To get the goal inequality in (3.3), it is sufficient to choose $\kappa > 0$ and $\varepsilon \in (0,\varepsilon_0)$ small enough in (3.9) such that

$$g(\varepsilon) := C\varepsilon^{-\kappa}\left(\frac{1}{\theta_1^{\log_2\left(\frac{1}{\varepsilon}\right)-1}}\right)^{\frac{1}{s}} < \frac{1}{2},$$

which finishes the proof. With the following presentation,

$$g(\varepsilon) := C\theta_1^{\frac{1}{s}}\left(\varepsilon^{-s\kappa}\theta_1^{\log_2\varepsilon}\right)^{\frac{1}{s}} = C\theta_1^{\frac{1}{s}}\left(\varepsilon^{\log_2\theta_1 - s\kappa}\right)^{\frac{1}{s}},$$

we remark that the choices of κ and ε are possible. Indeed, one just needs to choose

$$0 < \kappa < \frac{\log_2 \theta_1}{s}$$

and therefore $\lim_{\varepsilon \to 0^+} g\left(\varepsilon\right) = 0$. □

In the rest of the paper, we study the gradient estimate of weak solutions in the generalized Morrey spaces.

Theorem 3.2. *Let Ω be an open bounded domain in $\mathbb{R}^n$ such that $\partial\Omega$ is C^{1,σ^+} for some $\sigma^+ \in [\sigma, 1]$. Assume that $u \in W_0^{1,\Phi}\left(\Omega\right)$ is a weak solution to (1.6) under assumptions (1.3) and (1.7) with given data $F \in W^{1,\Phi}\left(\Omega\right)$. Let $s \in \left(0, \infty\right)$ and $\Psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function satisfying*

$$\Psi\left(x, 2\varrho\right) \le \gamma_0 \Psi\left(x, \varrho\right), \text{ for every } x \in \Omega \text{ and } 0 < \varrho < \operatorname{diam}(\Omega), \tag{3.10}$$

for some constant $\gamma_0 \in \left(1, 2^n\right)$. Then, for every $\beta \in \left[0, n\right)$ and $s \in \left(0, \infty\right)$, there holds

$$\left\|\mathbf{M}_\beta\left(\Phi(x, \nabla u)\right)\right\|_{\mathcal{M}^{s,\Psi}(\Omega)} \le C\left\|\mathbf{M}_\beta(\Phi(x, F))\right\|_{\mathcal{M}^{s,\Psi}(\Omega)}, \tag{3.11}$$

where the constant C depends on s, γ_0, $data$.

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Proof. For every $\beta \in \left[0, n\right)$, $s \in \left(0, \infty\right)$ and $\omega \in \mathbf{A}_\infty$, thanks to Theorem 3.1 with $t = s$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \left|\chi_\Omega \mathbf{M}_\beta\left(\Phi(x, \nabla u)\right)(\xi)\right|^s \omega(\xi) d\xi \le C \int_{\mathbb{R}^n} \left|\chi_\Omega \mathbf{M}_\beta(\Phi(x, F))(\xi)\right|^s \omega(\xi) d\xi. \tag{3.12}$$

Given $\delta \in \left(0, 1\right)$, $x \in \Omega$ and $0 < \varrho < \operatorname{diam}(\Omega)$, we further set $\mu := \chi_{B_\varrho(x)}$. Then, the following estimate holds true

$$\chi_{\Omega_\varrho(x)}(\xi) \le \mu(\xi) \le \mathbf{M}\mu(\xi) \le (\mathbf{M}\mu)^\delta(\xi) \le 1, \tag{3.13}$$

for every $\xi \in \mathbb{R}^n$. Therefore, we get that

$$\begin{aligned} T &:= \frac{1}{\Psi(x, \varrho)} \int_{\Omega_\varrho(x)} \left|\mathbf{M}_\beta\left(\Phi(x, \nabla u)\right)(\xi)\right|^s d\xi \\ &= \frac{1}{\Psi(x, \varrho)} \int_{\mathbb{R}^n} \left|\chi_\Omega \mathbf{M}_\beta\left(\Phi(x, \nabla u)\right)(\xi)\right|^s \mu(\xi) d\xi \\ &\le \frac{1}{\Psi(x, \varrho)} \int_{\mathbb{R}^n} \left|\chi_\Omega \mathbf{M}_\beta\left(\Phi(x, \nabla u)\right)(\xi)\right|^s (\mathbf{M}\mu)^\delta(\xi) d\xi. \end{aligned} \tag{3.14}$$

Thanks to Proposition 2 in Coifman and Rochberg (1980), one $(\mathbf{M}\mu)^{\delta} \in \mathbf{A}_1 \subset \mathbf{A}_{\infty}$, which allows us to deduce from (3.12) and (3.14) that

$$T \leq \frac{C}{\Psi(x,\varrho)} \int_{\Omega_{\varrho}(x)} \left| \mathbf{M}_{\beta}(\Phi(x,F))(\xi) \right|^{s} (\mathbf{M}\mu)^{\delta}(\xi) d\xi. \tag{3.15}$$

At this time, using the following dyadic decomposition of $\mathbb{R}^n$ by

$$\mathbb{R}^n = B_{2\varrho}(x) \cup \left(\bigcup_{k=1}^{\infty} B_{2^{k+1}\varrho}(x) \setminus B_{2^k\varrho}(x) \right),$$

the estimate (3.15) can be rewritten as below:

$$\begin{aligned} T \leq & \frac{C}{\Psi(x,\varrho)} \int_{B_{2\varrho}(x)} \left| \chi_{\Omega} \mathbf{M}_{\beta}(\Phi(x,F))(\xi) \right|^{s} (\mathbf{M}\mu)^{\delta}(\xi) d\xi \\ & + \frac{C}{\Psi(x,\varrho)} \sum_{k=1}^{\infty} \int_{B_{2^{k+1}\varrho}(x) \setminus B_{2^k\varrho}(x)} \left| \chi_{\Omega} \mathbf{M}_{\beta}(\Phi(x,F))(\xi) \right|^{s} (\mathbf{M}\mu)^{\delta}(\xi) d\xi. \end{aligned} \tag{3.16}$$

For every $\xi \in B_{2^{k+1}\varrho}(x) \setminus B_{2^k\varrho}(x)$, we have

$$2^k \varrho \leq \left| \xi - x \right| < 2^{k+1} \varrho.$$

Therefore, if $0 < r \leq (2^k - 1)\varrho$ then $B_r(\xi) \cap B_{\varrho}(x) = \emptyset$. It enables us to estimate

$$\begin{aligned} \mathbf{M}\mu(\xi) &= \sup_{r>0} \frac{1}{\mathcal{L}^n\left(B_r(\xi)\right)} \int_{B_r(\xi)} \left| \chi_{B_{\varrho}(x)}(\zeta) \right| d\zeta = \sup_{r>0} \frac{\mathcal{L}^n\left(B_r(\xi) \cap B_{\varrho}(x)\right)}{\mathcal{L}^n\left(B_r(\xi)\right)} \\ &= \sup_{r>(2^k-1)\varrho} \frac{\mathcal{L}^n\left(B_r(\xi) \cap B_{\varrho}(x)\right)}{\mathcal{L}^n\left(B_r(\xi)\right)} \\ &= \max\left\{ \sup_{(2^k-1)\varrho<r<(2^{k+1}+1)\varrho} \frac{\mathcal{L}^n\left(B_r(\xi) \cap B_{\varrho}(x)\right)}{\mathcal{L}^n\left(B_r(\xi)\right)}; \sup_{r\geq(2^{k+1}+1)\varrho} \frac{\mathcal{L}^n\left(B_{\varrho}(x)\right)}{\mathcal{L}^n\left(B_r(\xi)\right)} \right\}, \end{aligned}$$

which leads to

$$\frac{1}{(2^{k+1}+1)^n} \leq \mathbf{M}\mu(\xi) \leq \frac{1}{(2^k-1)^n}.$$

For this reason, we may approximate as follows:

$$\mathbf{M}\mu(\xi) \sim \frac{1}{2^{kn}} \text{ for every } \xi \in B_{2^{k+1}\varrho}(x) \setminus B_{2^k\varrho}(x).$$

Hence, applying the preceding estimates and taking into account (3.13), we conclude from (3.16) that

$$\begin{aligned} T &\le \frac{C}{\Psi(x,\varrho)}\int_{B_{2\varrho}(x)}\left|\chi_\Omega \mathbf{M}_\beta(\Phi(x,F))(\xi)\right|^s d\xi \\ &\qquad + C\sum_{k=1}^{\infty}\frac{1}{\Psi(x,\varrho)}\frac{1}{2^{kn\delta}}\int_{B_{2^{k+1}\varrho}(x)}\left|\chi_\Omega \mathbf{M}_\beta(\Phi(x,F))(\xi)\right|^s d\xi \\ &\le \frac{C\gamma_0}{\Psi(x,2\varrho)}\int_{B_{2\varrho}(x)}\left|\chi_\Omega \mathbf{M}_\beta(\Phi(x,F))(\xi)\right|^s d\xi \\ &\qquad + \sum_{k=1}^{\infty}\frac{C\gamma_0^{k+1}}{2^{kn\delta}}\frac{1}{\Psi(x,2^{k+1}\varrho)}\int_{B_{2^{k+1}\varrho}(x)}\left|\chi_\Omega \mathbf{M}_\beta(\Phi(x,F))(\xi)\right|^s d\xi \\ &\le C\gamma_0\left[1+\sum_{k=1}^{\infty}\left(\frac{\gamma_0}{2^{n\delta}}\right)^k\right]\left\|\mathbf{M}_\beta(\Phi(x,F))\right\|^s_{\mathcal{M}^{s,\Psi}(\Omega)}. \end{aligned} \tag{3.17}$$

Under the assumption $\gamma_0 \in \left(1, 2^n\right)$, one can choose $\delta \in \left(0,1\right)$ such that $\frac{\gamma_0}{2^{n\delta}} < 1$. For this reason, the series in (3.17) is convergent. In particular, there holds

$$1+\sum_{k=1}^{\infty}\left(\frac{\gamma_0}{2^{n\delta}}\right)^k = \frac{1}{1-\frac{\gamma_0}{2^{n\delta}}}.$$

The statement of inequality (3.11) is thus complete by taking the supremum of T in (3.17) for all $x \in \Omega$ and $0 < \varrho < \mathrm{diam}(\Omega)$. ☐

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KẾT QUẢ CHÍNH QUY NGHIỆM CHO PHƯƠNG TRÌNH DẠNG *(p,q)*-LAPLACE TRONG KHÔNG GIAN MORREY TỔNG QUÁT

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TÓM TẮT

Bài toán elliptic tựa tuyến tính không đồng nhất có nguồn gốc từ bài toán cực tiểu phiếm hàm tích phân năng lượng không tiêu chuẩn, được ứng dụng nhiều trong các ngành khoa học như Cơ học chất lỏng, Vật lí và bài toán đàn hồi phi tuyến. Một ví dụ điển hình cho lớp bài toán này là phương trình (p,q)-Laplace. Trong bài báo này, chúng tôi thiết lập các đánh giá gradient ứng với toán tử cực đại cấp phân số cho một lớp bài toán dạng (p,q)-Laplace trong không gian Morrey tổng quát. Kết quả chính quy toàn cục được chứng minh qua hai bước. Ở bước đầu tiên, chúng tôi xây dựng đánh giá gradient trong không gian Lorentz có trọng. Kết quả chính quy trong không gian Morrey được chứng minh trong bước thứ hai.

Từ khóa: không gian Morrey tổng quát; bài toán elliptic không đồng nhất; phương trình *(p,q)*-Laplace; chính quy nghiệm; không gian Lorentz có trọng