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Research Article

ON THE COMPARISON OF ISHIKAWA-TYPE ITERATIVE PROCESSES FOR CONTRACTION MAPPINGS IN BANACH SPACES WITH GRAPHS

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ABSTRACT

Numerous studies have examined the convergence of iterative processes with graphs to a common fixed point of contraction mappings. However, research comparing the convergence rates of these processes remains limited. This paper addresses this gap by analyzing the convergence rates of several Ishikawa-type iterative processes to a common fixed point of contraction mappings in Banach spaces with graphs. We propose sufficient conditions to determine the relative speed of convergence between iteration processes. Our work extends recent findings on the comparison of convergence rates between two-step and three-step iteration sequences, offering improved parameter range assumptions. Notably, this study introduces a novel approach to formulating optimal hypotheses for comparing convergence rates of general iterative processes.

Keywords: Banach spaces with graphs; contraction mappings; convergence rate; Ishikawa-type iterative processes

1. Introduction

Fixed point theory has been a subject of mathematical inquiry for approximately a century and continues to attract significant attention from contemporary mathematicians. One of the basic results in this theory is the Banach contraction principle, which confirms the existence of the unique fixed point $p \in \mathbb{X}$ of a contraction mapping $T : \mathbb{X} \to \mathbb{X}$ with a non-empty complete metric space $(\mathbb{X}, d)$. Furthermore, this fixed point can be approximated by the following Picard iterative sequence

$$\begin{cases} u_1 \in \mathbb{X}, \\ u_{n+1} = T(u_n), n \geq 1. \end{cases} \tag{1.1}$$

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More precisely, if T admits the contraction coefficient $\alpha \in [0,1)$ (i.e. $d(T(x),T(y)) \le \alpha d(x,y)$ for every $x,y \in \mathbb{X}$) then one can prove that this sequence converges to a fixed point $p \in \mathbb{X}$ with the error upper bound as below

$$d(u_{n+1},p) \le \alpha^n d(u_1,p), \quad \text{for every } n \ge 1.$$

We remind that the Banach contraction mapping principle can be stated in the special case when $\mathbb{X}$ in (1.1) is replaced by a non-empty convex set in Banach spaces. This theory is strongly developed and then applied in many fields such as functional analysis and partial differential equations. However, in many applications, it may not be possible to construct a Picard iterative sequence corresponding to a contraction mapping T as in (1.1). To overcome this difficulty, it is necessary to consider several different iterative processes which may be more complicated than the Picard sequence in (1.1). One of the first iterative sequences to be improved was the Mann iterative sequence in Dotson (1970) and Mann (1953), defined by

$$\begin{cases} u_1 \in \Omega, \\ u_{n+1} = (1-a_n)u_n + a_n T(u_n), \ n \ge 1. \end{cases} \tag{1.2}$$

Here we assume that the contraction mapping T from Ω into itself, where Ω is a non-empty convex set in the given Banach space $(\mathbb{X}, \|\cdot\|)$. Moreover, the real parameter sequences $(a_n)_{n\ge1}$ belongs to the interval [0,1]. Then, a new problem that arises is to compare the convergence rates between the Mann iteration (1.2) and the Picard iteration (1.1). More information about this is a study by Berinde (2004). For the same reason, more and more iterative sequences with two or three-step iterations are being investigated. The first two-step iteration was introduced by Ishikawa (1974) and Kalinde and Rhoades (1992) with the convergence of the following procedure being studied:

$$\begin{cases} u_1 \in \Omega, \\ u_{n+1} = (1-a_n)u_n + a_n T\left((1-b_n)u_n + b_n T(u_n)\right), \ n \ge 1, \end{cases} \tag{1.3}$$

It is well-known as the Ishikawa iteration, where $(b_n)_{n\ge1} \subset [0,1]$. This sequence was then modified by replacing T in (1.3) by two contraction mappings T_1, T_2 at two different steps respectively. There are several ways to modify the Ishikawa iteration. For instance, Tripak, (2016) considered the following iteration:

$$\begin{cases} u_1 \in \Omega, \\ v_n = (1-b_n)u_n + b_n T_1(u_n), \\ u_{n+1} = (1-a_n)u_n + a_n T_2(v_n), \ n \ge 1. \end{cases} \tag{1.4}$$

Then, other authors studied two generalized types of the Ishikawa iteration as

$$\begin{cases} u_1 \in \Omega, \\ v_n = (1-b_n)u_n + b_n T_1(u_n), \\ u_{n+1} = (1-a_n)T_1(u_n) + a_n T_2(v_n),\ n \geq 1 \end{cases} \tag{1.5}$$

in a study by Suparatulatorn et al. (2018), and

$$\begin{cases} u_1 \in \Omega, \\ v_n = (1-b_n)u_n + b_n T_1(u_n), \\ u_{n+1} = (1-a_n)T_1(v_n) + a_n T_2(v_n),\ n \geq 1, \end{cases} \tag{1.6}$$

in another study by Thianwan and Yambangwai (2019). It is worth mentioning that these iterative processes were considered by Aleomraninejad et al. (2012) and Jachymski (2008) for $\mathcal{G}$-contractive and $\mathcal{G}$-nonexpansive mappings in the Banach space $\mathbb{X}$ with a directed graph $\mathcal{G}$. Recently, Nguyen and Nguyen (2020) investigated the convergence of the iteration (1.6) to the common fixed point of two mappings T_1, T_2. On the other hand, they also proposed sufficient conditions on the parameter sequences (a_n) and (b_n) so that the iterations in (1.5) and (1.6) converge faster than the previous iterative sequence in (1.4). We now describe in detail the results related to this comparison. Let us assume that p is a common fixed point of two contraction mappings T_1, T_2 with the contraction factors $\alpha_1, \alpha_2 \in [0,1)$ respectively. Under the following assumptions

$$0 < \varepsilon \leq a_n \leq \delta < \frac{1-\alpha}{1+\alpha},\ \text{for } \alpha = \max\{\alpha_1, \alpha_2\},\ \text{and } 0 \leq b_n \leq 1, \tag{1.7}$$

Nguyen and Nguyen (2020) proved that both iterations in (1.5) and (1.6) converge faster than the previous one in (1.4). In the present paper, we prove that this result still holds under some weaker assumptions than (1.7). In particular, we will replace assumptions in (1.7) by large ranges of (a_n) and (b_n) as below

$$0 < \varepsilon \leq a_n \leq \delta \ \text{ and } \lambda \leq b_n \leq 1, \tag{1.8}$$

for some ε > 0, $\lambda \in [0,1]$ and suitable values of δ. For example, under the following additional assumption

$$\delta < \frac{1-\alpha_1(1-\lambda+\alpha_1\lambda)}{1+(2\alpha_2-\alpha_1)(1-\lambda+\alpha_1\lambda)}, \tag{1.9}$$

we show that the sequence (1.6) converges faster than (1.4) and the sequence (1.5) if provided

$$\delta < \frac{1-\alpha_1}{1-\alpha_1+2\alpha_2(1-\lambda+\alpha_1\lambda)}. \tag{1.10}$$

These results are stated in detail in Theorem 3.2. We emphasize that both upper bounds in (1.9) and (1.10) are strictly bigger than $\frac{1-\alpha}{1+\alpha}$ in many situations of given coefficients $\lambda, \alpha_1, \alpha_2$. Actually, our results are new even in the special case $\lambda = 0$, since

$$\frac{1-\alpha_1}{1-\alpha_1+2\alpha_2} > \frac{1-\alpha}{1+\alpha},$$

whenever $\alpha_1 \neq \alpha_2$. In other words, our results under assumptions (1.9) or (1.10) are non-trivial improvements with respect to the results by Nguyen and Nguyen (2020) under assumptions (1.7). Similarly, we next improve the similar results by Nguyen and Nguyen (2020) for the three-step iterative processes.

In numerical analysis, iterative processes are employed to compute fixed points of functions, which have diverse applications in science and engineering. A primary objective in enhancing the convergence rates of these iteration sequences to fixed points is to optimize computational efficiency and improve the precision of results. Using graphs in studying the convergence rates of iteration sequences has been applied in many recent studies (Nguyen & Nguyen, 2020; Tripak, 2016; Thianwan & Yambangwai, 2019). They show a visual representation of the convergence rates of these sequences. By performing calculations for specific sequences, researchers have generated corresponding graphs, enabling objective predictions about convergence rates based on these visual representations. Moreover, these graphs have been utilized as empirical evidence to support comparisons of convergence rates. A notable example is found in [8, Example 3.7], where the authors employed graphical illustrations to compare the convergence rates among several three-step iterative processes. This approach served as the inspiration for our current investigation into this topic.

The method employed in this study relies on relatively elementary calculations, focusing on more precise estimations of the upper bounds established by Nguyen and Nguyen (2020). The significant aspect of this approach lies in its potential to generate novel concepts for constructing optimal upper and lower bounds. These bounds, in turn, facilitate the derivation of comparative results regarding the convergence rates of iterative processes. In fact, we believe that if one can find optimal bounds depending on (a_n) and (b_n) then one can relax the constraints $a_n \in [\varepsilon, \delta]$ and $b_n \in [\lambda, 1]$ in (1.8). We will discuss this point in the future works.

The paper is structured as follows: Section 2 introduces key notations and definitions of contraction mappings with graphs, which are utilized throughout the study. Sections 3 and 4 present the main theorems, along with detailed proofs and discussions of the principal findings. The final section offers conclusions and outlines potential avenues for future research.

2. Notation and basic definitions

We first recall some well-known definitions by Jachymski (2008) and Nguyen and Nguyen (2020). Let Ω be a non-empty subset of a Banach space $(\mathbb{X}, \|\cdot\|)$ and Δ_Ω denote the diagonal of the Cartesian product $\Omega \times \Omega,$ that means

$$\Delta_\Omega = \{(u,u) : u \in \Omega\}.$$

Consider a directed graph $\mathcal{G}$ such that the set $V_{\mathcal{G}}$ of its vertices coincides with Ω, and the set $E_{\mathcal{G}}$ of its edges contains all loops, i.e., $V_{\mathcal{G}} = \Omega$ and $E_{\mathcal{G}} \supseteq \Delta_{\Omega}$. We assume $\mathcal{G}$ has no parallel edges, so we can identify $\mathcal{G}$ with the pair $(V_{\mathcal{G}}, E_{\mathcal{G}})$. Moreover, we assume that $E_{\mathcal{G}}$ is coordinate-convex, which means

$$(p,(1-t)u+tv) \in E_{\mathcal{G}} \text{ and } ((1-t)u+tv, p) \in E_{\mathcal{G}},$$

for every $(p,u),(p,v),(u,p),(v,p) \in E_{\mathcal{G}}$, and $t \in [0,1]$.

Definition 2.1. Let us consider a mapping $T : \Omega \to \Omega$.

i) We say that T preserves edges of $\mathcal{G}$ if $(T(u),T(v)) \in E_{\mathcal{G}}$ for every $(u,v) \in E_{\mathcal{G}}$.

ii) The mapping T is said to be $\mathcal{G}$-nonexpansive if there exists $\alpha \in [0,1]$ such that

$$\|T(u)-T(v)\| \le \alpha \|u-v\|, \quad \text{for all } (u,v) \in E_{\mathcal{G}}. \tag{2.1}$$

In this case, α is called the $\mathcal{G}$-nonexpansive factor of T. Moreover, if (2.1) holds for some $\alpha < 1$, we say that T is a $\mathcal{G}$-contraction with a contraction factor α.

Definition 2.2. Let (u_n) and (x_n) be two sequences in Ω. Assume that both sequences (u_n) and (x_n) converge to p in $\mathbb{X}$. We say that (u_n) converges to p faster than (x_n) if

$$\lim_{n\to\infty} \frac{\|u_n - p\|}{\|x_n - p\|} = 0. \tag{2.2}$$

According to this definition, to compare the convergence rate between two iterative processes (u_n) and (x_n), we need to calculate the limit of the following ratio

$$\mathcal{R}(u_n, x_n, p) = \frac{\|u_n - p\|}{\|x_n - p\|}. \tag{2.3}$$

3. Results

3.1. Comparison of two-step iterative processes

In this section, we will compare the convergence speed of Ishikawa-type iterative processes defined in (1.4), (1.5), and (1.6). Let us change the notation to distinguish these iterations. From now on, we denote by $(x_n),(w_n)$ and (u_n) the sequences determined by (1.4), (1.5), and (1.6) respectively, this means

$$\begin{cases} y_n = (1-b_n)x_n + b_n T_1(x_n), \\ x_{n+1} = (1-a_n)x_n + a_n T_2(y_n), \; n \ge 1, \end{cases} \tag{3.1}$$

$$\begin{cases} z_n = (1-b_n)w_n + b_n T_1(w_n), \\ w_{n+1} = (1-a_n)T_1(w_n) + a_n T_2(z_n), \; n \ge 1, \end{cases} \tag{3.2}$$

and

$$\begin{cases} v_n = (1-b_n)u_n + b_n T_1(u_n), \\ u_{n+1} = (1-a_n)T_1(v_n) + a_n T_2(v_n),\ n \geq 1. \end{cases} \tag{3.3}$$

Here, without loss of generality, we may assume that $x_1 = w_1 = u_1$. Before stating the main results of the two-step iterative processes, let us introduce some initial assumptions.

Assumption 3.1. Throughout this section, we always suppose that all of the following hypotheses hold.

i) $T_1, T_2 : \Omega \to \Omega$ preserve edges of $\mathcal{G}$.

ii) T_1, T_2 are $\mathcal{G}$-contractions with factors $\alpha_1, \alpha_2 \in [0,1)$ respectively.

iii) T_1, T_2 have a common fixed point $p \in \Omega$.

iv) Let $u_1 \in \Omega \setminus \{p\}$ such that $(u_1, p), (p, u_1) \in E_{\mathcal{G}}$.

v) Two parameter sequences (a_n) and (b_n) belong to the following ranges

$$0 < \varepsilon \leq a_n \leq \delta \text{ and } \lambda \leq b_n \leq 1, \tag{3.4}$$

for some constants ε, λ and δ.

Let us now state our first main result.

Theorem 3.2. Let us consider three iterative processes $(x_n), (w_n)$ and (u_n) defined in (3.1), (3.2), and (3.3) respectively, under Assumption 3.1. Suppose further that

$$\delta < \frac{1-\mu\alpha_1}{1+\mu(2\alpha_2-\alpha_1)}, \tag{3.5}$$

where μ is defined by

$$\mu := 1-\lambda+\alpha_1\lambda. \tag{3.6}$$

Then the sequence (u_n) converges to p faster than (x_n). Moreover, (w_n) also converges faster than (x_n) under the following additional condition

$$\delta < \frac{1-\alpha_1}{1-\alpha_1+2\mu\alpha_2}. \tag{3.7}$$

Proof. Thanks to Nguyen and Nguyen (2020) (Proposition 3.4), all sequences defined in (3.1), (3.2), and (3.3) are well-defined, i.e. (u_n, p) and $(p, u_n) \in E_{\mathcal{G}}$ for all $n \geq 1$. Let us first show that the sequence (u_n) converges to p faster than (x_n) in the sense of (2.2) in Definition 2.2. Indeed, since $T_1(p) = p$ then we can present $\|v_n - p\|$ in (3.3) as follows

$$\begin{aligned} \|v_n - p\| &= \|(1-b_n)u_n + b_n T_1(u_n) - (1-b_n)p - b_n T_1(p)\| \\ &\leq (1-b_n)\|u_n - p\| + b_n \|T_1(u_n) - T_1(p)\|, \end{aligned}$$

for every $n \geq 1$. Combining this with the fact that T_1 is $\mathcal{G}$-contraction, one obtains that

$$\|v_n - p\| \le (1-b_n)\|u_n - p\| + \alpha_1 b_n\|u_n - p\| = [1-(1-\alpha_1)b_n]\|u_n - p\|. \quad (3.8)$$

Similarly, in the second step of (3.3), since $T_1(p) = T_2(p) = p$ and T_1, T_2 are $\mathcal{G}$-contractions with corresponding factors α_1, α_2, it follows that

$$\begin{aligned}\|u_{n+1} - p\| &\le (1-a_n)\|T_1(v_n) - T_1(p)\| + a_n\|T_2(v_n) - T_2(p)\| \\ &\le [\alpha_1 + (\alpha_2 - \alpha_1)a_n]\|v_n - p\|.\end{aligned} \quad (3.9)$$

Substituting (3.8) into (3.9), we get that

$$\begin{aligned}\|u_{n+1} - p\| &\le [\alpha_1 + (\alpha_2 - \alpha_1)a_n][1-(1-\alpha_1)b_n]\|u_n - p\| \\ &\le \mu[\alpha_1 + (\alpha_2 - \alpha_1)a_n]\|u_n - p\|,\end{aligned} \quad (3.10)$$

where μ is given as in (3.6). We remark that the last inequality in (3.10) comes from the assumption $b_n \ge \lambda$ in (3.4).

We next consider the sequence (x_n) given by (3.1). On one hand, by similar computation, there holds

$$\begin{aligned}\|x_{n+1} - p\| &\le \left[1 - a_n + \alpha_2 a_n(1 - b_n + \alpha_1 b_n)\right]\|x_n - p\| \\ &\le \left[1-(1-\alpha_2)\varepsilon\right]\|x_n - p\|.\end{aligned} \quad (3.11)$$

It leads to $\|x_{n+1} - p\| \le \left[1-(1-\alpha_2)\varepsilon\right]^n\|x_1 - p\| \to 0$ as $n \to \infty$, which implies that (x_n) converges to p in $\mathbb{X}$. In the first step, assumption $\mathcal{G}$-contraction of T_1 gives us

$$\|y_n - p\| \le [1-(1-\alpha_1)b_n]\|x_n - p\|. \quad (3.12)$$

In the second step of (3.1), we estimate in a different way. More precisely, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-a_n)x_n + a_n T_2(y_n) - (1-a_n)p - a_n T_2(p)\| \\ &\ge (1-a_n)\|x_n - p\| - a_n\|T_2(y_n) - T_2(p)\| \\ &\ge (1-a_n)\|x_n - p\| - \alpha_2 a_n\|y_n - p\|,\end{aligned}$$

which by (3.12) deduces to

$$\begin{aligned}\|x_{n+1} - p\| &\ge \{1 - a_n - \alpha_2 a_n[1-(1-\alpha_1)b_n]\}\|x_n - p\| \\ &\ge \left(1 - a_n - \mu\alpha_2 a_n\right)\|x_n - p\|.\end{aligned} \quad (3.13)$$

Here, we note that $1 - a_n - \mu\alpha_2 a_n > 0$ under assumptions (3.4) and (3.5), since

$$\frac{1}{1+\mu\alpha_2} \ge \frac{1-\mu\alpha_1}{1+\mu(2\alpha_2 - \alpha_1)} > a_n.$$

Thanks to (3.10) and (3.13), it leads to

$$0 \le \mathcal{R}(u_{n+1}, x_{n+1}, p) \le \frac{\mu\alpha_1 + \mu(\alpha_2 - \alpha_1)a_n}{1-(1+\mu\alpha_2)a_n} \cdot \mathcal{R}(u_n, x_n, p), \quad (3.14)$$

for all $n \geq 1$, where the ratio $\mathcal{R}$ is defined by (2.3). To bind the factor on the right-hand side of (3.14), let us consider the function $f:[0,\delta] \to \mathbb{R}^+$ defined by

$$f(t) = \frac{\mu\alpha_1 + \mu(\alpha_2 - \alpha_1)t}{1-(1+\mu\alpha_2)t}, \quad t \in [0,\delta].$$

It is easy to check that

$$f'(t) = \frac{\mu\alpha_2(1+\mu\alpha_1)}{[1-(1+\mu\alpha_2)t]^2} > 0, \quad \text{for all } t \in [0,\delta].$$

This guarantees that f is non-decreasing on $[0,\delta]$. Moreover, combining this with assumption $(a_n) \subset [0,\delta]$, we always have $f(a_n) \leq f(\delta)$. For this reason, by (3.14), one gets that

$$0 \leq \mathcal{R}(u_{n+1}, x_{n+1}, p) \leq \left[f(\delta)\right]^n. \tag{3.15}$$

Finally, assumption (3.5) ensures that $f(\delta) < 1$, which leads to

$$\lim_{n\to\infty} \mathcal{R}(u_{n+1}, x_{n+1}, p) = 0$$

from (3.15). By Definition 2.2, (u_n) converges to p faster than (x_n).

Performing the same procedure as before for the sequence (3.2), one obtains that

$$0 \leq \mathcal{R}(w_{n+1}, x_{n+1}, p) \leq g(a_n)\mathcal{R}(w_n, x_n, p), \tag{3.16}$$

for all $n \geq 1$, where the function g is defined by

$$g(t) = \frac{\alpha_1 + (\mu\alpha_2 - \alpha_1)t}{1-(1+\mu\alpha_2)t}, \quad t \in [0,\delta].$$

Since $g'(t) > 0$, then one has $g(a_n) \leq g(\delta) < 1$ under assumption (3.7). Therefore, by (3.16), we get that

$$0 \leq \mathcal{R}(w_{n+1}, x_{n+1}, p) \leq \left[g(\delta)\right]^n \to 0 \text{ as } n \to \infty.$$

This allows us to conclude the second statement of this theorem. The proof is complete.

Remark 3.3. It is necessary to state Assumption 3.1.iv) $u_1 \neq p$, to avoid the case $x_n = w_n = p_n = p$, for all $n \geq 1$, then the sequences will have the same convergence rate.

Remark 3.4. Since $\mu \in [\alpha_1, 1]$, it is not difficult to check that

$$\frac{1-\mu\alpha_1}{1+\mu(2\alpha_2 - \alpha_1)} \geq \frac{1-\alpha_1}{1-\alpha_1+2\mu\alpha_2},$$

which confirms that the assumption (3.5) is weaker than (3.7).

Remark 3.5. We remark that if $a_n \equiv 0$ then $x_n \equiv x_1$ for all $n \geq 1$. Hence, assumption $a_n \geq \varepsilon$ is sufficient for the convergence of x_n to p. Two other sequences are even convergent without this assumption.

Remark 3.6. It can be seen that the upper bound in (3.11) is not sharp. Hence, if one can find a better upper bound here then one can relax the assumption $a_n \geq \varepsilon$. The idea is the same in inequalities (3.13) and (3.14). For this reason, we believe that the result in Theorem 3.2 can be improved by relaxing the assumption (3.4) of the two sequences (a_n) and (b_n). However, the calculations may be quite complicated, so we will discuss them in other articles.

3.2. Comparison of three-step iterative processes

The present section deals with three-step iterative processes. Let us fix $x_1 \in \Omega$, we define the sequence (x_n) as follows

$$\begin{cases} z_n = (1-c_n)x_n + c_n T_3(x_n), \\ y_n = (1-b_n)z_n + b_n T_2(z_n), \\ x_{n+1} = (1-a_n)y_n + a_n T_1(y_n),\ n \geq 1. \end{cases} \tag{3.17}$$

We show that with any influence of the contraction mapping to the second or third step of the iterative process, then the new one converges faster than (3.17) under suitable assumptions of parameter sequences. Let us first consider the sequence (u_n) defined by

$$\begin{cases} w_n = (1-c_n)u_n + c_n T_3(u_n), \\ v_n = (1-b_n)T_3(w_n) + b_n T_2(w_n), \\ u_{n+1} = (1-a_n)T_2(v_n) + a_n T_1(v_n),\ n \geq 1. \end{cases} \tag{3.18}$$

where $u_1 = x_1$. In the previous section, we introduce some initial assumptions.

Assumption 3.7. We suppose that all of the following hypotheses hold.

i) $T_1, T_2, T_3 : \Omega \to \Omega$ preserve edges of $\mathcal{G}$.

ii) T_1, T_2, T_3 are $\mathcal{G}-$contractions with factors $\alpha_1, \alpha_2, \alpha_3 \in [0,1)$ respectively.

iii) T_1, T_2, T_3 have a common fixed point $p \in \Omega$.

iv) Let $u_1 \in \Omega \setminus \{p\}$ such that $(u_1, p), (p, u_1) \in E_{\mathcal{G}}$.

v) Three parameter sequences $(a_n), (b_n)$ and (c_n) satisfy the following conditions

$$\varepsilon \leq \min\{a_n, b_n, c_n\} \leq \max\{a_n, b_n, c_n\} \leq \delta, \tag{3.19}$$

for some positive constants ε and δ.

Theorem 3.8. Under Assumption (3.17), let $\delta < \frac{1}{1+\alpha}$ with $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and satisfy the following condition

$$\frac{\alpha_2 + (\alpha_1 - \alpha_2)\delta}{1-(1+\alpha_1)\delta} \cdot \frac{\alpha_3 + (\alpha_2 - \alpha_3)\delta}{1-(1+\alpha_2)\delta} \cdot \frac{1-(1-\alpha_3)\delta}{1-(1+\alpha_3)\delta} < 1. \tag{3.20}$$

Then the sequence (u_n) in (3.18) converges to p faster than (x_n) in (3.17).

Proof. For the sequence (x_n) defined in (3.17), we can obtain two estimates as below

$$\| x_{n+1} - p \| \le (1 - a_n + \alpha_1 a_n)(1 - b_n + \alpha_2 b_n)(1 - c_n + \alpha_3 c_n) \| x_n - p \|, \tag{3.21}$$

and

$$\| x_{n+1} - p \| \ge (1 - a_n - \alpha_1 a_n)(1 - b_n - \alpha_2 b_n)(1 - c_n - \alpha_3 c_n) \| x_n - p \|, \tag{3.22}$$

for every $n \ge 1$. By condition $\min\{a_n, b_n, c_n\} \ge \varepsilon$ in (3.19) and inequality (3.21), it allows us to arrive

$$\| x_{n+1} - p \| \le \left[1 - (1-\alpha)\varepsilon\right] \| x_n - p \| \le \left[1 - (1-\alpha)\varepsilon\right]^n \| x_1 - p \|.$$

This implies that (x_n) converges to p under the basic condition $\min\{a_n, b_n, c_n\} \ge \varepsilon$. Next, let us consider the sequence (u_n) in (3.18), there holds

$$\| u_{n+1} - p \| \le \left[\alpha_1 a_n + \alpha_2 (1 - a_n)\right]\left[\alpha_2 b_n + \alpha_3 (1 - b_n)\right](\alpha_3 c_n + 1 - c_n) \| u_n - p \|. \tag{3.23}$$

Combining two estimates in (3.22) and (3.23), it yields that

$$0 \le \mathcal{R}(u_{n+1}, x_{n+1}, p) \le F(a_n) G(b_n) H(c_n) \mathcal{R}(u_n, x_n, p), \tag{3.24}$$

Where $F, G, H : [0, \delta] \to \mathbb{R}^+$ are defined by

$$F(t) = \frac{\alpha_2 + (\alpha_1 - \alpha_2)t}{1 - (1 + \alpha_1)t}, \; G(t) = \frac{\alpha_3 + (\alpha_2 - \alpha_3)t}{1 - (1 + \alpha_2)t}, \quad H(t) = \frac{1 - (1 - \alpha_3)t}{1 - (1 + \alpha_3)t}.$$

These functions are non-decreasing, which by (3.24) and assumptions (3.19) imply to

$$0 \le \mathcal{R}(u_{n+1}, x_{n+1}, p) \le [F(\delta) G(\delta) H(\delta)]^n. \tag{3.25}$$

We recall moreover that $\mathcal{R}(u_1, x_1, p) = 1$ by choosing $u_1 = x_1$. Condition (3.20) is equivalent to $F(\delta) G(\delta) H(\delta) < 1$. For this reason, we conclude that (u_n) converges to p faster than (x_n).

Let us now discuss a special case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \in (0,1)$. In this case, from (3.20) and (3.25), we get

$$0 \le \mathcal{R}(u_{n+1}, x_{n+1}, p) \le [K(t)]^n,$$

where the function $K : \left[0, \frac{1}{1+\alpha}\right) \to \mathbb{R}^+$ is defined by

$$K(t) = \frac{\alpha^2 \left[1 - (1-\alpha)t\right]}{\left[1 - (1+\alpha)t\right]^3}.$$

It is clear that K is increasing and continuous in $\left[0, \frac{1}{1+\alpha}\right)$. Moreover, since

$$K(0) = \alpha^2 < 1, \text{ and } \lim_{t \to \left(\frac{1}{1+\alpha}\right)^-} K(t) = +\infty,$$

Then there exists a unique $t_0 \in \left(0, \frac{1}{1+\alpha}\right)$ satisfying $K(t_0) = 1$. As a consequence, for any $\delta \in [0, t_0)$, one has $K(\delta) < 1$, which leads to (3.20). In other words, the conclusion of Theorem (3.18) holds under assumption $a_n \le \delta$ for every $\delta < t_0$.

Let us now consider the corresponding hypothesis by Nguyen and Nguyen (2020), Proposition 2.1.6. With $\tilde{\delta} = \frac{1-\sqrt[3]{\alpha^2}}{1+\alpha}$, one has

$$K(\tilde{\delta}) = 1 - (1-\alpha)\tilde{\delta} < 1,$$

which implies that $\tilde{\delta} < t_0$. Nguyen and Nguyen (2020) with Proposition 2.1.6 showed that (u_n) converges faster than (x_n) under assumption $a_n \le \tilde{\delta}$. Roughly speaking, our result in this case is better than the corresponding one reported by Nguyen and Nguyen (2020).

With the similar techniques as in the proof of Theorem 3.18, we also obtain the comparisons between two other three-step iterative sequences, compared with the first one in (3.17).

Theorem 3.9. Under Assumption 3.17, let $\delta < \frac{1}{1+\alpha}$ with $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and satisfy the following condition

$$\frac{1-(1-\alpha_1)\delta}{1-(1+\alpha_1)\delta} \cdot \frac{\alpha_1(1-\delta) + \alpha_2\delta[1-(1-\alpha_3)\delta]}{[(1-(1+\alpha_2)\delta][1-(1+\alpha_3)\delta]} < 1. \tag{3.26}$$

Then the following sequence

$$\begin{cases} w_n = (1-c_n)u_n + c_n T_3(u_n), \\ v_n = (1-b_n)T_1(u_n) + b_n T_2(w_n), \\ u_{n+1} = (1-a_n)v_n + a_n T_1(v_n), \ n \ge 1, \end{cases}$$

converges faster than (x_n) in (3.17).

Theorem 3.10. Under Assumption 3.17, let $\delta < \frac{1}{1+\alpha}$ with $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and satisfy the following condition

$$\frac{\alpha_2 + (\alpha_1 - \alpha_2)\delta}{1-(1+\alpha_1)\delta} \cdot \frac{1-(1-\alpha_2)\delta}{1-(1+\alpha_2)\delta} \cdot \frac{1-(1-\alpha_3)\delta}{1-(1+\alpha_3)\delta} < 1. \tag{3.27}$$

Then sequence (u_n) defined by

$$\begin{cases} w_n = (1-c_n)u_n + c_n T_3(u_n), \\ v_n = (1-b_n)w_n + b_n T_2(w_n), \\ u_{n+1} = (1-a_n)T_2(v_n) + a_n T_1(v_n), \ n \ge 1, \end{cases}$$

converges faster than (x_n) in (3.17).

4. Conclusion

This study presents findings on the comparative convergence rates of two-step and three-step Ishikawa-type iterative algorithms. The results offer novel insights, even when applied to specific cases such as those examined by Nguyen and Nguyen (2020). Our research introduces innovative approaches to constructing optimal upper and lower bounds for iterative sequences, facilitating more straightforward comparisons of convergence rates. We intend to further develop these concepts in subsequent publications.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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SO SÁNH TỐC ĐỘ HỘI TỤ CỦA CÁC DÃY LẶP KIỂU ISHIKAWA CỦA CÁC ÁNH XẠ CO TRONG KHÔNG GIAN BANACH VỚI ĐỒ THỊ

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TÓM TẮT

Đã có nhiều kết quả về sự hội tụ của các dãy lặp đến điểm bất động chung của các ánh xạ co với đồ thị. Tuy nhiên, có rất ít kết quả liên quan đến việc so sánh tốc độ hội tụ của các dãy lặp này. Trong bài báo này, chúng tôi nghiên cứu về tốc độ hội tụ của một số dãy lặp kiểu Ishikawa đến điểm bất động chung của các ánh xạ co trong không gian Banach với đồ thị. Cụ thể hơn, chúng tôi đề xuất một số điều kiện đủ để đảm bảo một dãy lặp hội tụ nhanh hơn một dãy lặp khác. Công việc của chúng tôi đã cải thiện được một kết quả của một nghiên cứu gần đây liên quan đến việc so sánh tốc độ hội tụ của các dãy lặp hai và ba bước. Nhìn chung, các giả thiết chúng tôi xây dựng về khoảng của các dãy tham số tốt hơn các kết quả trước đó. Một điểm thú vị là bài báo của chúng tôi mở ra một ý tưởng mới về việc xây dựng một giả thiết tối ưu để có thể so sánh được tốc độ hội tụ của các dãy lặp tổng quát.

Từ khóa: không gian Banach với đồ thị; Ánh xạ co, tốc độ hội tụ; Dãy lặp kiểu Ishikawa