

Research Article

**SOME RESULTS ON THE 0TH FORMAL LOCAL COHOMOLOGY
WITH RESPECT TO A NON-MAXIMAL IDEAL***Tran Tuan Nam¹, Nguyen Hoang Huy Tu^{2,3*}*¹*Ho Chi Minh City University of Education, Viet Nam*²*University of Science, Vietnam National University Ho Chi Minh City, Vietnam*³*Basic Scientific Faculty, Foreign Trade University, Ho Chi Minh City Campus, Ho Chi Minh, Vietnam***Corresponding author: Nguyen Hoang Huy Tu – Email: nguyenhoanghuytu.cs2@ftu.edu.vn**Received: February 12, 2024; Revised: February 28, 2024; Accepted: March 22, 2024***ABSTRACT**

This paper studies the formal local cohomology with respect to non-maximal ideals. It is an extension of the formal local cohomology for maximal ideals by Schenzel (2007). By using the properties of inverse limits and I -adic topology, we have proved several crucial properties of the formal local cohomology of order 0 regarding non-maximal ideals. In particular, detailed computations were provided for the local cohomology of order 0 as regards non-maximal ideals. The equivalent properties of the formal local cohomology of order 0 regarding non-maximal ideals and associated primes of modules are rigorously proven.

Keywords: cohomology dimension; formal cohomology; local cohomology

Mathematics subject classification (2010): 13D45.

1. Introduction

Throughout this article, let R denote a commutative Noetherian ring with identity. Let I, J be ideals of R . To investigate the asymptotic behavior of the inverse system of local cohomology modules $\left\{ H_J^i(M/I^t M) \right\}_{t \in \mathbb{N}}$, Schenzel introduces the concept of formal cohomology. To be more specific, for an R -module M , the module

$$\mathcal{F}_{I,J}^i(M) := H^i \left(\varprojlim_t (C_{\underline{x}}^\bullet \otimes_R M/I^t M) \right)$$

is called the i -th I -formal local cohomology module of M with respect to J , where $\underline{x} = x_1, x_2, \dots, x_r$ is a system of elements such that $J = \sqrt{(x_1, x_2, \dots, x_r)R}$, and $C_{\underline{x}}^\bullet$ is the Čech

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complex of R with respect to \underline{x} . Schenzel (2007) [3.2], he show that there are the following short exact sequences

$$0 \longrightarrow \varprojlim^1 H_J^{i-1}(M/I^t M) \longrightarrow \mathcal{F}_{I,J}^i(M) \longrightarrow \varprojlim H_J^i(M/I^t M) \longrightarrow 0$$

for all $i \in \mathbb{Z}$. In particular, when R is a local ring with the maximal ideal \mathfrak{m} and $J = \mathfrak{m}$, we obtain isomorphisms

$$\mathcal{F}_{I,\mathfrak{m}}^i(M) \cong \varprojlim H_{\mathfrak{m}}^i(M/I^t M)$$

for a finitely generated R -module M (Schenzel, 2007). Most works on formal local cohomology concern this particular case (Asgharzadeh & Divaani-azar, 2011; Tran et al., 2022; Schenzel, 2007; Yan, 2014). This paper reports the results of formal local cohomology in the case that J is a non-maximal ideal.

The organization of our paper is as follows. Some preliminaries about inverse limits, formal local cohomology and its basic properties are given in Section 2. The last section is devoted to study some properties of the 0th formal local cohomology module under certain assumptions. Theorem 3.3 states that if M is a finitely generated module over a complete ring R with the I -adic topology, then there is an isomorphism

$$\mathcal{F}_{I,J}^0(M) \cong \bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle)$$

Theorems 3.4 and 3.5 prove equivalent properties of the formal local cohomology of order 0 with respect to non-maximal ideals and associated primes of modules.

2. Preliminaries

In this paper, it is assumed that R is a commutative Noetherian ring. We recall the concept of formal local cohomology (Schenzel, 2007)). Let I, J be ideals of R and M an R -module. Let $C_{\underline{x}}^{\bullet}$ denote the Čech complex of R with respect to $\underline{x} = x_1, x_2, \dots, x_r$ where $J = \sqrt{(x_1, x_2, \dots, x_r)R}$. Now the system of modules $\{M/I^t M\}_{t \in \mathbb{N}}$ induces an inverse system of R -complexes $\{C_{\underline{x}}^{\bullet} \otimes_R M/I^t M\}_{t \in \mathbb{N}}$.

Definition 2.1. With the previous notation, the i -th I -formal local cohomology module of M with respect to J , denoted by $\mathcal{F}_{I,J}^i(M)$, is defined by

$$\mathcal{F}_{I,\mathfrak{m}}^i(M) := H^i \left(\varprojlim^1 (C_{\underline{x}}^{\bullet} \otimes_R M/I^t M) \right).$$

When there is no doubt on I , we simply say formal cohomology with respect to J .

The following theorem shows the relation of formal local cohomology with respect to the inverse limit and its derived function of certain local cohomology modules. For the sake of completeness, we provide a proof.

Proposition 2.2. (See also Schenzel, 2007). Let M be an R -module. There are the following short exact sequences

$$0 \longrightarrow \varprojlim^1 H_J^{i-1}(M / I^t M) \longrightarrow \mathcal{F}_{I,J}^i(M) \longrightarrow \varprojlim^i H_J^i(M / I^t M) \longrightarrow 0$$

for all $i \in \mathbb{Z}$.

Proof. We first note that $\{M / I^t M\}_{t \in \mathbb{N}}$ is a surjective system. Then the inverse system of R -complexes $\{C_{\underline{x}}^\bullet \otimes_R M / I^t M\}_{t \in \mathbb{N}}$ is also a surjective system. Analysis similar to that in [11, Page 84], we have the short exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim^1 H^{i-1}(C_{\underline{x}}^\bullet \otimes_R M / I^t M) &\longrightarrow \mathcal{F}_{I,J}^i(M) \\ &\longrightarrow \varprojlim^i H^i(C_{\underline{x}}^\bullet \otimes_R M / I^t M) \longrightarrow 0 \end{aligned}$$

for all $i \in \mathbb{Z}$. On the other hand, there are isomorphisms

$$H^i(C_{\underline{x}}^\bullet \otimes_R M / I^t M) \cong H_J^i(M / I^t M)$$

for all integer t . Hence we obtain the short exact sequence as required. \square

From Definition 2.1, we obtain the property that equivalent ideal topologies define isomorphic formal cohomology modules.

Lemma 2.3. (Tran & Nguyen 2023). Let M denote an R -module. Let $\{M_t\}_{t \in \mathbb{N}}$ be a decreasing family of submodules of M . Suppose that there exists an integer t_0 such that $I^{t+t_0} N \subseteq M_t$ and $M_{t+t_0} \subseteq I^t M$ for all $t \geq 0$. It is equivalent to say that the topology induced by $\{M_t\}_{t \in \mathbb{N}}$ is equivalent to the I -adic topology on M . Then there are isomorphisms

$$\mathcal{F}_{I,J}^i(M) \cong H^i\left(\varprojlim^i (C_{\underline{x}}^\bullet \otimes_R M / M_t)\right)$$

for all $i \in \mathbb{Z}$.

Let (R, \mathfrak{m}) be a local ring. Let $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ denote a short exact sequence of finitely generated R -modules. In [Schenzel P. (2007), Theorem 3.11], Schenzel proves that there is a long exact sequence

$$\dots \longrightarrow \mathcal{F}_{I,\mathfrak{m}}^i(N) \longrightarrow \mathcal{F}_{I,\mathfrak{m}}^i(M) \longrightarrow \mathcal{F}_{I,\mathfrak{m}}^i(L) \longrightarrow \dots$$

This is an important tool to investigate formal local cohomology with respect to a maximal ideal. We prove a similar result for a non-maximal ideal as found in Tran & Nguyen (2023).

Theorem 2.4. (Tran & Nguyen 2023). Let

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

be a short exact sequence of R -modules. Suppose that N is finitely generated R -module. For ideals I, J of R , there is a long exact sequence

$$\dots \longrightarrow \mathcal{F}_{I,J}^i(N) \longrightarrow \mathcal{F}_{I,J}^i(M) \longrightarrow \mathcal{F}_{I,J}^i(L) \longrightarrow \dots$$

Let us recall the Mittag-Leffler condition for the exactness of the inverse limits: An inverse system of R -modules, which is indexed by \mathbb{N} , is said to satisfy the Mittag-Leffler condition if for each n , the decreasing family $\{\phi_{n'n}(M_{n'}) \subseteq M_n : n' \geq n\}$ of submodules of M_n is stationary. In other words, for each n , there is an $n_0 \geq n$, such that for all $n', n'' \geq n_0$, we have $\phi_{n'n}(M_{n'}) = \phi_{n''n}(M_{n''})$ as submodule of M_n . If $\{M_n, \phi_{n'n}\}$ is an inverse system of Artinian modules, then it satisfies the Mittag-Leffler condition.

Lemma 2.5. (Hartshorne, 1977). *Let*

$$0 \longrightarrow \{N_n\} \longrightarrow \{M_n\} \longrightarrow \{L_n\} \longrightarrow 0$$

be a short exact sequence of the inverse system of modules. Then

(i) *If $\{M_n\}$ satisfies the Mittag-Leffler condition, then so does $\{L_n\}$.*

(ii) *If $\{N_n\}$ satisfies the Mittag-Leffler condition, then sequence of inverse limits*

$$0 \longrightarrow \varprojlim_n N_n \longrightarrow \varprojlim_n M_n \longrightarrow \varprojlim_n L_n \longrightarrow 0$$

is exact.

3. On the 0th formal local cohomology modules

Let R be a Noetherian ring. Let I, J denote ideals of R . By the view of Proposition 2.2, the 0th formal local cohomology module with respect to J is isomorphic to the inverse limit of certain local cohomology modules.

Lemma 3.1. *Let M be an R -module. Then*

$$\mathcal{F}_{I,J}^0(M) \cong \varprojlim_t H_J^0(M / I^t M).$$

Proof. It immediately follows from Proposition 2.2. \square

From the previous result, we can calculate the 0th formal local cohomology module in some settings.

Lemma 3.2. *Let M be a finitely generated R -module. Suppose that*

$$\text{Supp}(M) \cap V(I) \subseteq V(J).$$

Then $\mathcal{F}_{I,J}^i(M) = 0$ for all $i \neq 0$ and

$$\mathcal{F}_{I,J}^0(M) \cong \Lambda_I(M)$$

where $\Lambda_I(M)$ is the I -adic completion of M .

Proof. Let t be an integer. We have that

$$\text{Supp}(M / I^t M) = \text{Supp}(M) \cap V(I) \subseteq V(J).$$

Since R is the Noetherian ring and M is a finitely generated R -module, there exists an integer n such that $J^n(M/I^tM) = 0$. In other words, M/I^tM is an J -torsion module so $H_J^i(M/I^tM) = 0$ for all $i \neq 0$ and $H_J^0(M/I^tM) \cong M/I^tM$. Taking the inverse limit, we get an isomorphism

$$\varprojlim_t H_J^0(M/I^tM) \cong \Lambda_I(M)$$

and $\varprojlim_t H_J^i(M/I^tM) = 0$ for all $i \neq 0$. Now the claim follows from the previous proposition. \square

Let M be a finitely generated R -module. For an R -submodule N of M , there is an increasing family of submodules of N

$$N :_M I \subseteq N :_M I^2 \subseteq \dots \subseteq N :_M I^n \dots$$

Therefore, let $N :_M \langle I \rangle$ denote the maximal module in that chain.

Next, the following result provides the precise computation of the 0th formal local cohomology module with respect to J when R is I -adic complete (the natural homomorphism $R \rightarrow \Lambda_I(R)$ is an isomorphism).

Theorem 3.3. *Let M be a finitely generated R -module. Suppose that R is complete in the I -adic topology. Then there is an isomorphism*

$$\mathcal{F}_{I,J}^0(M) \cong \bigcap_{t \in \mathbb{N}} (I^tM :_M \langle J \rangle)$$

Proof. First, we note that

$$I^tM :_M \langle J \rangle / I^tM \cong H_J^0(M/I^tM)$$

for all integer t . Moreover, there are the natural short exact sequences

$$0 \longrightarrow I^tM \longrightarrow I^tM :_M \langle J \rangle \longrightarrow I^tM :_M \langle J \rangle / I^tM \longrightarrow 0.$$

Taking the inverse limit with index t and the notice that $\bigcap_{t \in \mathbb{N}} I^tM = 0$ by the Krull intersection theorem, we obtain

$$0 \longrightarrow \bigcap_{t \in \mathbb{N}} (I^tM :_M \langle J \rangle) \xrightarrow{\varphi} \varprojlim_t H_J^0(M/I^tM).$$

Now we will show that φ is surjective. Indeed, let

$$\{x_t + I^tM\} \in \varprojlim_t (I^tM :_M \langle J \rangle / I^tM).$$

where $x_t \in I^tM :_M \langle J \rangle$ for all t . Next, consider $\varprojlim_t (I^tM :_M \langle J \rangle / I^tM)$ as a submodule of the I -adic completion $\Lambda_I(M)$. Since R is I -adic complete and M is finitely generated R -module, M is complete in its I -adic topology. Therefore, there exists an element $z \in M$

such that $z - x_t \in I^t M$ for all integer t . It is obvious that $z \in \bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle)$ and $\varphi(z) = \{x_t + I^t M\}$. Finally, the claim follows from Lemma 3.1. \square

As an application of Theorem 3.3, we show the vanishing of the 0th formal local cohomology module with respect to J .

Theorem 3.4. *Let M denote a finitely generated R -module. Suppose that R is complete in the I -adic topology. Then the following statements are equivalent.*

i) $\mathcal{F}_{I,J}^0(M) = 0$.

ii) For any prime ideal $\mathfrak{p} \in \text{Ass}_R(M)$, $J^m \not\subseteq I + \mathfrak{p}$ for all integer m .

In addition, if (R, \mathfrak{m}) is a complete local ring, then (i) and (ii) are equivalent to

iii) For any t , there is an integer $s(t)$ such that $I^{s(t)} M :_M \langle J \rangle \subseteq I^t M$.

Proof. (i) \Rightarrow (ii). Let $\mathfrak{p} = (0 :_R x)$ be an associated prime of R -module M such that $J^m \subseteq I + \mathfrak{p}$ for an integer m . Obviously, x is a non-zero element of M so, by Theorem 3.3, it is enough to show that for all integer n there is an integer m such that $J^m x \subseteq I^n M$. Now, let n be an arbitrary integer. By the assumption, we have that $J^m \subseteq I + (0 :_R x)$. In other words, $J^m x \subseteq IM$. Thus, $J^{mm} x \subseteq I^n M$.

(ii) \Rightarrow (i). By the previous theorem, there is a non-zero element $x \in \bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle)$. Therefore, $J^m \subseteq I + (0 :_R x)$ for an integer m . Since $\text{Ass}_R(Rx) \subseteq \text{Ass}_R(M)$, there is $\mathfrak{p} \in \text{Ass}_R(M)$ such that $(0 :_R x) \subseteq \mathfrak{p}$. Hence

$$J^m \subseteq I + (0 :_R x) \subseteq I + \mathfrak{p}.$$

(i) \Rightarrow (iii). For any integer t , we have the short exact sequence

$$0 \longrightarrow H_J^0(M / I^t M) \longrightarrow M / I^t M \longrightarrow M / (I^t M :_M \langle J \rangle) \longrightarrow 0$$

Note that $I^t M :_M \langle J \rangle / I^t M \cong H_J^0(M / I^t M)$. Now, $H_J^0(M / I^t M)$ is semi-discrete linear compact because R is complete in \mathfrak{m} -adic topology and $H_J^0(M / I^t M)$ is finitely generated R -module. Therefore, the sequence

$$0 \longrightarrow \mathcal{F}_{I,J}^0(M) \longrightarrow \varprojlim_t M / I^t M \longrightarrow \varprojlim_t M / (I^t M :_M \langle J \rangle) \longrightarrow 0$$

is exact by Theorem 3.3 and (Jensen, 1972), 7.1). Thus, the exact sequence implies that $\{I^t M\}_{t \in \mathbb{N}}$ and $\{I^t M :_M \langle J \rangle\}_{t \in \mathbb{N}}$ induces the same topology on M . Hence for any t , there is an integer $s(t)$ such that $I^{s(t)} M :_M \langle J \rangle \subseteq I^t M$.

(iii) \Rightarrow (i). In virtue Theorem 3.3, it suffices to prove $\bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle)$ is zero. For any integer t ,

$$I^{s(t)} M \subseteq I^{s(t)} M :_M \langle J \rangle \subseteq I^t M$$

These inclusions induce

$$\bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle) = \bigcap_{t \in \mathbb{N}} I^t M$$

and the second module is equal to zero by the Krull intersection theorem. It finishes the proof. \square

The hypothesis that R is complete, the I -adic topology can be omitted in (i) \Rightarrow (ii) of Theorem 3.4. Indeed, from the proof Theorem 3.3, there is an injection

$$\bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle) \hookrightarrow \mathcal{F}_{I,J}^0(M).$$

Therefore, we can consider $\bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle)$ as a submodule of the formal local cohomology module $\mathcal{F}_{I,J}^0(M)$. Furthermore, it is easy to see that $\text{Ass}_R \left(\bigcap_{t \in \mathbb{N}} (I^t M :_M \langle J \rangle) \right)$ contains all prime ideal $\mathfrak{p} \in \text{Ass}_R(M)$ such that $J^m \subseteq I + \mathfrak{p}$ for an integer m .

Theorem 3.5. *Let (R, \mathfrak{m}) be a local ring. Let M denote a finitely generated R -module. Then*

$$\mathcal{F}_{I,\mathfrak{m}}^0(M) \cong \bigcap_{t \in \mathbb{N}} (I^t \hat{M} :_{\hat{M}} \langle \hat{\mathfrak{m}} \rangle)$$

and the following statements are equivalent.

- i) $\mathcal{F}_{I,\mathfrak{m}}^0(M) = 0$.
- ii) For any prime ideal $P \in \text{Ass}_{\hat{R}}(\hat{M})$, $\dim \hat{R} / (\hat{I}R + P) > 0$.
- iii) For any t , there is an integer $s(t)$ such that $I^{s(t)} \hat{M} :_{\hat{M}} \langle \hat{\mathfrak{m}} \rangle \subseteq I^t \hat{M}$.

Proof. First, for each t , $H_{\mathfrak{m}}^0(M / I^t M)$ has a natural structure of an \hat{R} module and, in this case, the natural homomorphism

$$H_{\mathfrak{m}}^0(M / I^t M) \rightarrow H_{\mathfrak{m}}^0(M / I^t M) \otimes_R \hat{R}$$

is an isomorphism. Now, the flatness of \hat{R} over R induces that

$$H_{\mathfrak{m}}^0(M / I^t M) \cong H_{\hat{\mathfrak{m}}}^0(\hat{M} / I^t \hat{M})$$

Thus, the inverse limit provides that

$$\mathcal{F}_{I,\mathfrak{m}}^0(M) \cong \mathcal{F}_{\hat{I}R, \hat{\mathfrak{m}}}^0(\hat{M})$$

Hence the claim follows from Theorem 3.3 and Theorem 3.4. \square

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MỘT SỐ KẾT QUẢ VỀ ĐỐI ĐỒNG ĐIỀU HÌNH THỨC BẬC 0 ỨNG VỚI IDEAN KHÔNG TỐI ĐẠI

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TÓM TẮT

Bài báo này nghiên cứu về đối đồng điều địa phương hình thức ứng với ideal không tối đại. Đây là một mở rộng của đối đồng điều địa phương hình thức ứng với ideal tối đại của Schenzel (2007). Bằng cách sử dụng các tính chất của giới hạn ngược và tô pô I -adic chúng tôi đã chứng minh được một số tính chất quan trọng của đối đồng điều địa phương hình thức ứng bậc 0 ứng với ideal không tối đại. Cụ thể, chúng tôi đưa ra các tính toán chi tiết cho đối đồng điều địa phương hình thức bậc 0 ứng với ideal không tối đại. Các tính chất tương đương của đối đồng điều địa phương hình thức bậc 0 ứng với ideal không tối đại và các ideal nguyên tố liên kết với môđun cũng được chứng minh chặt chẽ.

Từ khóa: chiều đối đồng điều; đối đồng điều hình thức; đối đồng điều địa phương