ASYMPTOTIC FORMULAS FOR THE FORCES AND TORQUES OF TWO CLOSE PARTICLES IN A STOKES FLUID

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ABSTRACT

The main goal of this paper is to establish the asymptotic formulas of forces and torques exerted by two closed spherical particles with prescribed velocities moving in a viscous fluid. Our proof is based on the decomposition into some more simple motions after a singular-regular decomposition of forces (torques) in the inner and outer region of expansions.

Keywords: Stokes equations, asymptotic formula, Fluid Dynamics.

TÓM TẮT

Công thức tiệm cận cho lực và ngẫu lực của hai quả cầu gần nhau trong dòng chảy Stokes

Mục tiêu chính của bài báo này là thiết lập công thức tiệm cận của hai đại lượng lực và ngẫu lực, được tạo ra bởi hai quả cầu rất gần nhau, chuyển động với vận tốc xác định trong một dòng chảy nhớt. Chứng minh của chúng tôi dựa trên việc phân chia thành nhiều chuyển động đơn giản hơn, sau khi đã tách lực (ngẫu lực) thành tổng hai lực và dựa vào các khai triển trên miền bên trong và miền bên ngoài.

Từ khóa: phương trình Stokes, công thức tiệm cận, cơ học chất lỏng.

Many of numerical methods were proposed these last years to compute the hydrodynamic interactions between rigid spheres in a Stokes fluid, such as Brady *et al.* (1987, 1988), Cichocki *et al.* (1994), Ladd (1988), or recently Nguyen (2013), Lefebvre *et al.* (2014)... The motivation comes from the model of nano-scale swimmers, such as sperm cells, swimming bacteria or unicellular algae or recent advances in creating artificial nanoscale swimmers designed to deliver medication from nanosized medical devices. Some numerical simulations of these model were also studied by Alouges *et al.* (2008, 2013), Lefebvre *et al.* (2009). To do this, we have to compute the hydrodynamic forces and torques generated by a viscous fluid on each particle. The well known difficulty in such numerical simulations is take into account the singular lubrication forces exerted by the fluid remaining in the gap between close particles. So it is very interesting if we have an asymptotic formula of force of two close particles in this case.

The sequel is organized as follows. In Section 1, we describe the setting of the problem and the main result. The proof of this result is presented in Section 2, 3 and 4.

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1. Setting of the problem

We consider two non intersecting particles immersed in a viscous fluid. For simplicity, the particles are identical balls B_1, B_2 with radius 1 and centers z_1, z_2, respectively. We assume that these closed balls do not intersect and that the fluid fills the rest of the space. The fluid occupies the domain $\Omega := R^3 \setminus \left(B_1 \cup B_2\right)$.

We assume moreover that the fluid inertia effects are negligible compared to the viscosity (i.e. the Reynolds number is very small $\mathrm{Re} << 1$) so that the velocity u and the pressure p solve the stationary Stokes equations in the fluid domain,

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla . u = 0 & \text{in } \Omega. \end{cases} \tag{1}$$

On the surfaces of the particles, we consider a no-slip condition,

$$u = u_i \quad \text{on } \partial B_i, \quad i = 1, 2,$$

where the velocity u_i corresponds to a rigid displacement. It is characterized by the velocity U_i at the center z_i of the ball B_i and by the angular velocity w_i $\left(U_i,\ w_i \in R^3\right)$,

$$u_i\left(x\right) = U_i + w_i \times (x - z_i), \quad \text{for } i = 1, 2. \tag{2}$$

where "$\times$" denotes the cross product in R^3.

We are interested in solutions u which decay at infinity, i.e., which fulfills

$$u\left(x\right) \to 0, \quad as\ \left|x\right| \to +\infty.$$

We note that the existence and uniqueness of a solution to (1) is classical in the Hilbert space

$$D^{1,2}\left(\Omega\right) := \left\{ u \in D'\left(\Omega, R^3\right):\ \nabla u \in L^2\left(\Omega\right),\ \frac{u}{\sqrt{1+\left|x\right|^2}} \in L^2\left(\Omega\right),\ \nabla . u = 0 \text{ in } \Omega \right\},$$

endowed with the scalar product
$$\left(u, v\right)_D = \int_\Omega \nabla u : \nabla v.$$

The surface density of force exerted on the fluid at some point x of the surface ∂B_i is given by

$$f_i(x) = \left(\nabla u + \nabla u^T - pId\right).n_i,$$

where n_i denotes the exterior normal on the surface of the i-est particle B_i. The total force and total torque exerted by the particle B_i on the fluid are given by the following formulas,

$$F_i = \int_{\partial B_i} f_i(x)dS, \qquad \tau_i = \int_{\partial B_i} (x - z_i) \times f_i(x)dS, \tag{3}$$

where n is the unit normal to the surface and dS is the element of area of the surface.

The main goal of this paper is to give a complete proof of the following result:

The main result

Assume that F and τ are respectively the force and torque exerted by the fluid on the first particle and are given by the formula (3). Let us set

$$V = \frac{U_1 - U_2}{2} + \frac{w_1 - w_2}{2} \times \frac{z_2 - z_1}{2} \text{ and } \omega = \frac{w_1 - w_2}{2}.$$

Then $F \approx F^{asympt}$ and $\tau \approx \tau^{asympt}$ *as d tends to 0, where*

$F^{asympt} = \left(F_1^{asympt}, F_2^{asympt}, F_3^{asympt}\right)$ and $\tau^{asympt} = \left(\tau_1^{asympt}, \tau_2^{asympt}, \tau_3^{asympt}\right)$ *with*

$$F_1^{asympt} = 2\pi V_1 \ln d + O\left(d^0\right),$$

$$F_2^{asympt} = 2\pi V_2 \ln d + O\left(d^0\right),$$

$$F_3^{asympt} = -3\pi V_3 d^{-1} + O\left(\ln d\right),$$

$$\tau_1^{asympt} = \left(-2\pi V_2 + \frac{6\pi}{5}\omega_1\right)\ln d + O\left(d^0\right),$$

$$\tau_2^{asympt} = \left(2\pi V_1 + \frac{6\pi}{5}\omega_2\right)\ln d + O\left(d^0\right),$$

$$\tau_3^{asympt} = O\left(d^0\right).$$

The proof of this result is given by three steps. In the first step, we decompose the force (torque) into two parts: regular part and singular part. The regular part is order of d^0, so we just need to establish the asymptotic formula for the singular part. In the next step, we expand the velocity u and pressure p in the power series of the distance d. Then, we form the equations of the leading terms based on a decomposition into inner and outer region of expansion. In the last step, by linearity of the equations, we decompose the total force (torque) as a sum of several forces (torques) which correspond to simple motions. The detailed process is described in the three next sections.

2. Decomposition in regular and singular parts

We decompose the rigid displacements (2) as follows:

$$u_1(x) = \frac{U_1+U_2}{2} + \frac{U_1-U_2}{2} + \frac{w_1+w_2}{2} \times (x-z_1) + \frac{w_1-w_2}{2} \times (x-z_1),$$
$$u_2(x) = \frac{U_1+U_2}{2} + \frac{U_2-U_1}{2} + \frac{w_1+w_2}{2} \times (x-z_2) + \frac{w_2-w_1}{2} \times (x-z_2).$$

To lighten motion, let us introduce the mean values:

$$\overline{U} := \frac{U_1+U_2}{2},\ \ \overline{\omega} := \frac{w_1+w_2}{2},\ \ \overline{z} := \frac{z_1+z_2}{2}.$$

The two rigid velocities u_1 and u_2 can be decomposed as sums of singular and regular part as follows

$$u_i = u_i^r(x) + u_i^s(x), \quad \text{for } i = 1,2, \tag{4}$$

where

$$u^r = \overline{U} + \overline{\omega} \times (x - \overline{z}),$$
$$u_1^s = \frac{U_1-U_2}{2} + \overline{\omega} \times \frac{z_2-z_1}{2} + \frac{w_1-w_2}{2} \times (x-z_1),$$
$$u_2^s = \frac{U_2-U_1}{2} - \overline{\omega} \times \frac{z_2-z_1}{2} + \frac{w_2-w_1}{2} \times (x-z_2).$$

It is convenient to set $V = \frac{U_1-U_2}{2} + \overline{\omega} \times \frac{z_2-z_1}{2}$ and $\omega = \frac{w_1-w_2}{2}$, then the singular parts of the velocities rewrite as

$$u_1^s = V + \omega \times (x-z_1),$$
$$u_2^s = -V - \omega \times (x-z_2).$$

By linearity, the corresponding force densities are given by $f_i = f^r + f_i^s$.

We note that in the decomposition (4), since the first term u^r corresponds to a rigid displacement of the object formed by the two balls, we do not expect it to lead to a singular force density. We have $f^r = O(1)$. Without loss the general, we assume $u^r = 0$, that is:

$$u_i(x) = \pm\, V \pm\, \omega \times (x - z_i). \tag{5}$$

For simplicity, we just consider the total force F and torque τ exerted by the fluid on the first particle. The total force and torque on the other particle are obtained by symmetry. Recall that F and τ are given by

$$F = \int_{\partial B_1} f_1 dS, \qquad \tau = \int_{\partial B_1} n \times f_1 dS.$$

3. Inner and outer region of expansion

It is known that the expansions of velocity u and pressure p are singular in terms of the distance d. So we consider two regions of expansion. An outer region of expansion is defined using the outer variables (x_1, x_2, x_3) in euclidean coordinates. In these coordinate systems, the velocity u and the pressure p have the forms

$$u(x) = (u_1(x), u_2(x), u_3(x)) \quad \text{and} \quad p = p(x),$$

with $x = (x_1, x_2, x_3)$. The system of equations (1) is valid in this region. For some very small distances d tends to 0, the particles are almost in contact and the point ofcontact will be a singular point for the flow. So it is necessary to build a new coordinates system for inner region of expansion.

The variations in the inner region of expansion are described using the inner variables $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$:

$$\bar{x}' = d^{-1/2} x', \quad \bar{x}_3 = d^{-1} x_3,$$

where $\bar{x}' = (\bar{x}_1, \bar{x}_2)$. In this coordinate system, the velocity and pressure fields are given by

$$\bar{u}_1(\bar{x}) = d^{1/2-k} u_1(x), \qquad \bar{u}_2(\bar{x}) = d^{1/2-k} u_2(x),$$
$$\bar{u}_3(\bar{x}) = d^{-k} u_3(x), \qquad \bar{p}(\bar{x}) = d^{2-k} p(x),$$

where $x = (x_1, x_2, x_3), \bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and k is a real constant which is defined later.

We have:

$$\nabla_x u = \begin{pmatrix} d^{k-3/2} \nabla_{\bar{x}} \bar{u}_1 \\ d^{k-3/2} \nabla_{\bar{x}} \bar{u}_2 \\ d^{k-3/2} \nabla_{\bar{x}} \bar{u}_3 \end{pmatrix} \begin{pmatrix} d^{1/2} & 0 & 0 \\ 0 & d^{1/2} & 0 \\ 0 & 0 & d^{1/2} \end{pmatrix}.$$

From the scaling relations between inner and outer variables we have:

$$\nabla_x p = \left(d^{k-5/2} \partial_{\bar{x}_1} \bar{p}, d^{k-5/2} \partial_{\bar{x}_2} \bar{p}, d^{k-5/2} \partial_{\bar{x}_3} \bar{p} \right)^T,$$

$$\Delta_x u_1 = d^{k-3/2} \nabla^2_{\bar{x}'} \bar{u}_1 + d^{k-5/2} \frac{\partial^2 \bar{u}_1}{\partial \bar{x}_3^2},$$

$$\Delta_x u_2 = d^{k-3/2} \nabla^2_{\bar{x}'} \bar{u}_2 + d^{k-5/2} \frac{\partial^2 \bar{u}_2}{\partial \bar{x}_3^2},$$

$$\Delta_x u_3 = d^{k-1} \nabla^2_{\bar{x}'} \bar{u}_3 + d^{k-2} \frac{\partial^2 \bar{u}_3}{\partial \bar{x}_3^2}.$$

We then expand formally $\bar{u}$ and $\bar{p}$ on the forms

$$\bar{u} = \bar{u}^0 + d\bar{u}^1 + d^2 \bar{u}^2 + \ldots$$

$$\bar{p} = \bar{p}^0 + d\bar{p}^1 + d^2 \bar{p}^2 + \ldots$$

Plugging these expansions in (1) and identifying the terms of the power series in d^i, $i = k-5/2, k-2, k-3/2, \ldots$, we obtain that the flow field $\left(\bar{u}^0, \bar{p}^0\right)$ satisfies:

$$\begin{aligned} &\frac{\partial^2 \bar{u}_1^0}{\partial \bar{x}_3^2} - \frac{\partial^2 \bar{p}^0}{\partial \bar{x}_1} = 0, \quad \frac{\partial^2 \bar{u}_2^0}{\partial \bar{x}_3^2} - \frac{\partial^2 \bar{p}^0}{\partial \bar{x}_2} = 0, \quad \frac{\partial^2 \bar{p}^0}{\partial \bar{x}_3} = 0, \\ &\frac{\partial^2 \bar{u}^0}{\partial \bar{x}_1} + \frac{\partial^2 \bar{u}^0}{\partial \bar{x}_2} + \frac{\partial^2 \bar{u}^0}{\partial \bar{x}_3} = 0 \end{aligned} \tag{6}$$

The flow field $\left(\bar{u}^1, \bar{p}^1\right)$ satisfies:

$$\frac{\partial^2 \bar{u}_1^1}{\partial \bar{x}_3^2} - \frac{\partial^2 \bar{p}^1}{\partial \bar{x}_1} = -\nabla^2_{\bar{x}'} \bar{u}_1^0, \quad \frac{\partial^2 \bar{u}_2^1}{\partial \bar{x}_3^2} - \frac{\partial^2 \bar{p}^1}{\partial \bar{x}_2} = -\nabla^2_{\bar{x}'} \bar{u}_2^0,$$

$$\frac{\partial^2 \bar{p}^1}{\partial \bar{x}_3} = \frac{\partial^2 \bar{u}_3^0}{\partial \bar{x}_3^2}, \quad \frac{\partial^2 \bar{u}^1}{\partial \bar{x}_1} + \frac{\partial^2 \bar{u}^1}{\partial \bar{x}_2} + \frac{\partial^2 \bar{u}^1}{\partial \bar{x}_3} = 0.$$

There is no difficulty in principle which prevents us from now proceeding to calculate further terms in the expansion, but for the purpose of the analysis, we only need to consider the leading order given by (6).

Now, it is convenient to change variables:

$$y_1 = \bar{x}_1, \quad y_2 = \bar{x}_2, \quad y_3 = \bar{x}_3 + \frac{1}{2} + \frac{1}{2}\left(\bar{x}_1^2 + \bar{x}_2^2\right).$$

The surfaces of the particles near the contact point respectively satisfy

$$\partial B_1^{inner}: \quad y_3 = O(d),$$
$$\partial B_1^{inner}: \quad y_3 = 1 + y_1^2 + y_2^2 + O(d) := h(y_1, y_2) + O(d).$$

For $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and $y = (y_1, y_2, y_3)$, let us set $v(y) := \bar{u}_0(\bar{x}); \quad q(y) := \bar{p}_0(\bar{x}),$

with these new variables and $v(y) = (v_1(y), v_2(y), v_3(y))$, the equation (6) reads as follows:

$$\frac{\partial^2 v_1}{\partial y_3^2} - \frac{\partial q}{\partial y_1} = 0, \quad \frac{\partial^2 v_2}{\partial y_3^2} - \frac{\partial q}{\partial y_2} = 0, \frac{\partial q}{\partial y_3} = 0, \tag{7}$$

$$\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{\partial v_3}{\partial y_3} + y_1 \frac{\partial v_1}{\partial y_3} + y_2 \frac{\partial v_2}{\partial y_3} = 0. \tag{8}$$

Let us consider the force F and torque τ exerted by the fluid on the first particle. The force and torque on the other particle can be obtained by symmetry. In inner variables, the unit normal n to the surface and the element of area of the surface dS are given by:

$$n = \left(d^{1/2} y_1 + O(d), d^{1/2} y_2 + O(d), 1 + O(d)\right), \qquad dS = d.dy_1 dy_2 (1 + O(d)).$$

We note that the singular terms of force and torque are contained in the leading term of the inner expansion. Moreover the asymptotic formulas of force F^{asympt} and torque τ^{asympt} at small gaps are only generated on the area of surface around the contact points. Hence we may compute F^{asympt} and τ^{asympt} on the small surface $\left\{x \in \partial B_1 : x_1^2 + x_2^2 \le \varepsilon^2\right\}$, where ε is a small real number. In inner variables, this surface becomes:

$$S_\varepsilon = \left\{y \in \partial B_1^{inner} : \; y_1^2 + y_2^2 \le d^{-1}\varepsilon^2\right\}.$$

Then we obtain

$$\begin{aligned}
F_1^{asympt} &= d^{k-1/2} \int_{S_\varepsilon} \left(-y_1 q + \frac{\partial v_1}{\partial y_3}\right) dy_1 dy_2 + O(d^k), \\
F_2^{asympt} &= d^{k-1/2} \int_{S_\varepsilon} \left(-y_2 q + \frac{\partial v_2}{\partial y_3}\right) dy_1 dy_2 + O(d^k), \\
F_3^{asympt} &= d^{k-1} \int_{S_\varepsilon} \left(-q\right) dy_1 dy_2 + O(d^k),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\tau_1^{asympt} &= d^{k-1/2}\int_{S_\varepsilon}\left(-\frac{\partial v_2}{\partial y_3}\right)dy_1dy_2 + O(d^k),\\
\tau_2^{asympt} &= d^{k-1/2}\int_{S_\varepsilon}\left(\frac{\partial v_1}{\partial y_3}\right)dy_1dy_2 + O(d^k),\\
\tau_3^{asympt} &= d^{k}\int_{S_\varepsilon}\left(y_1\frac{\partial v_2}{\partial y_3} - y_2\frac{\partial v_1}{\partial y_3}\right)dy_1dy_2 + O(d^{k+1/2}).
\end{aligned}\tag{10}$$

The boundary conditions for v in inner variables reads

$$\begin{aligned}
v_1 &= V_1d^{1/2-k} - \omega_3y_2d^{1-k} + O(d^{3/2-k}),\\
v_2 &= V_2d^{1/2-k} + \omega_3y_1d^{1-k} + O(d^{3/2-k}),\\
v_3 &= V_3d^{-k} + (\omega_1y_2 - \omega_2y_1)d^{1/2-k} \quad \text{on} \quad \partial B_1^{inner},
\end{aligned}\tag{11}$$

and

$$\begin{aligned}
v_1 &= -V_1d^{1/2-k} + \omega_3y_2d^{1-k} + O(d^{3/2-k}),\\
v_2 &= -V_2d^{1/2-k} - \omega_3y_1d^{1-k} + O(d^{3/2-k}),\\
v_3 &= -V_3d^{-k} - (\omega_1y_2 - \omega_2y_1)d^{1/2-k} \quad \text{on} \quad \partial B_2^{inner},
\end{aligned}\tag{12}$$

where V and ω in (5) have components $V = (V_1, V_2, V_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$.

4. Asymptotic formulas for total force and torque

Since (v,q) linearly depends on V and ω, we may decompose the velocity field (v,q) in three parts

$$v = v_A + v_B + v_C, \quad q = q_A + q_B + q_C,$$

where the first part (v_A, q_A) is the flow resulting from the translational motion of surfaces along the vertical axis, the second part (v_B, q_B) is the flow resulting from the tangential and rolling motion of surfaces and the last part (v_C, q_C) is the folw resulting from the rotational motion of surfaces about normal. More precisely, three flow fields (v_A, q_A), (v_B, q_B) and (v_C, q_C) satisfy (7), (8) with the following boundary conditions

$$(v_A)_1 = (v_B)_2 = 0, \ \ (v_A)_3 = \pm V_3d^{-k} \quad \text{on} \quad \partial B_i^{inner}, \tag{13}$$

$$\begin{aligned}
&(v_B)_1 = \pm V_1d^{1/2-k} + O(d^{3/2-k}), \ \ (v_B)_2 = \pm V_2d^{1/2-k} + O(d^{3/2-k}),\\
&(v_B)_3 = \pm(\omega_1y_2 - \omega_2y_1)d^{1/2-k} \quad \text{on} \ \partial B_i^{inner},
\end{aligned}\tag{14}$$

$$(v_C)_1 = \mp\omega_3y_2d^{1-k}, \ (v_C)_2 = \pm\omega_3y_1, \ (v_C)_3 = 0 \quad \text{on} \ \partial B_i^{inner}. \tag{15}$$

From the boundary condition (13), (14) and (15) we deduce that in order to calculate (v_A, q_A) one must take $k=0$, $k=1/2$ for (v_B, q_B) and $k=1$ for (v_C, q_C). Next we build the symptotic formulas of force and torque which are correspondingly decomposed as

$$F^{asympt} = F_A + F_B + F_C, \quad \tau^{asympt} = \tau_A + \tau_B + \tau_C.$$

4.1. Translation motion of spheres

Since (v_A, q_A) satisfies (7), (8) we obtain

$$q_A = q_A(y_1 y_2), \ \ (v_A)_1 = \frac{1}{2}\frac{\partial q_A}{\partial y_1} + Ay_3 + C, \ \ (v_A)_2 = \frac{1}{2}\frac{\partial q_A}{\partial y_2} + By_3 + D,$$

where A, B, C and D are arbitrary functions of y_1 and y_2. These terms may be determined from the above boundary conditions of v_A as

$$A = -\frac{1}{2}\frac{\partial q_A}{\partial y_1} h(y_1, y_2), \ \ B = -\frac{1}{2}\frac{\partial q_A}{\partial y_2} h(y_1, y_2), \ \ C = D = 0. \tag{16}$$

Hence, $(v_A)_1$ and $(v_A)_2$ become

$$(v_A)_1 = \frac{1}{2}\left(y_3^2 - y_3 h\right)\frac{\partial q_A}{\partial y_1}, \qquad (v_B)_2 = \frac{1}{2}\left(y_3^2 - y_3 h\right)\frac{\partial q_A}{\partial y_2}. \tag{17}$$

Substituting the expressions of $(v_A)_1$, $(v_A)_2$ given by (7) into (8) and then integrating with respect to y_3 we get

$$(v_A)_3 = -\frac{1}{6}\left(\frac{\partial^2 q_A}{\partial y_1^2} + \frac{\partial^2 q_A}{\partial y_2^2}\right)y_3^3 - \frac{1}{2}\left(\frac{\partial A}{\partial y_1} + \frac{\partial B}{\partial y_2} + y_1\frac{\partial q_A}{\partial y_1} + y_2\frac{\partial q_A}{\partial y_2}\right)y_3^2 - (Ay_1 + By_2)y_3 + E, \tag{18}$$

where E is a function of y_1 and y_2.

Since $(v_A)_3 = -V_3$ on the surface $y_3 = 0$ in the limit of $d \downarrow 0$, it follows that $y = -V_3$. Similarly, since $(v_A)_3 = 1$ on the surface $y_3 = h$, we get:

$$-V_3 = -\frac{1}{6}\left(\frac{\partial^2 q_A}{\partial y_1^2} + \frac{\partial^2 q_A}{\partial y_2^2}\right)h^3 - \frac{1}{2}\left(\frac{\partial A}{\partial y_1} + \frac{\partial B}{\partial y_2} + y_1\frac{\partial q_A}{\partial y_1} + y_2\frac{\partial q_A}{\partial y_2}\right)h^2 - (Ay_1 + By_2)h + E.$$

After substituting the values of A and B from (16) into the above equality and then simplifying we obtain:

$$\nabla \cdot \left(h^3 \nabla q_A\right) = -24V_3. \tag{19}$$

In order to solve this equation, we use the polar coordinates

$$y_1 = \bar{r}\cos\theta, \qquad y_2 = \bar{r}\sin\theta,$$

so that the equation (19) takes the form

$$\bar{r}^{2}\frac{\partial^2 q_A}{\partial \bar{r}^2} + \frac{\partial^2 q_A}{\partial \theta^2} + \left(\bar{r} + \frac{6r^3}{1+\bar{r}^2}\right)\frac{\partial q_A}{\partial \bar{r}} = \frac{-24V_3\bar{r}^2}{\left(1+\bar{r}^2\right)^3}. \tag{20}$$

If we assume that q_A is of order $\bar{r}^n$ as $\bar{r} \to \infty$, then $(v_A)_1, (v_A)_2$ are $O(\bar{r}^{n-1})$ and $(v_A)_3$ is of the form $-1+O(\bar{r}^n)$ as $\bar{r} \to \infty$. By expressing these qualities in outer variables and noting that the pressure and velocity in the outer region of expansion cannot contain any terms which tend to infinity as d tends to 0, this shows that $n \le -4$. Hence $q_A = O(\bar{r}^{-4})$ as $\bar{r} \to \infty$.

The solution of (20) which satisfies the above condition could be

$$q_A = \frac{3}{(1+\bar{r}^2)^2} + O(d). \tag{21}$$

The error term of order d in the expression of q_A arises from the fact that the expressions given in the boundary conditions have an error of order d.

From (9) and (10), the symptotic formulas F_A and τ_A generated from the flow field (v_A, q_A) are given by

$$\left(F_A\right)_1 = d^{-1/2}\int_{S_\varepsilon}\left(-y_1 q_A + \frac{\partial (v_A)_1}{\partial y_3}\right)dy_1 dy_2 + O(d^0),$$

$$\left(F_A\right)_2 = d^{-1/2}\int_{S_\varepsilon}\left(-y_2 q_A + \frac{\partial (v_A)_2}{\partial y_3}\right)dy_1 dy_2 + O(d^0),$$

$$\left(F_A\right)_1 = d^{-1}\int_{S_\varepsilon}\left(-q_A\right)dy_1 dy_2 + O(d^0),$$

and

$$\left(\tau_A\right)_1 = d^{-1/2}\int_{S_\varepsilon}\left(-\frac{\partial (v_A)_2}{\partial y_3}\right)dy_1 dy_2 + O(d^0),$$

$$\left(\tau_A\right)_2 = d^{-1/2}\int_{S_\varepsilon}\left(\frac{\partial (v_A)_1}{\partial y_3}\right)dy_1 dy_2 + O(d^0),$$

$$\left(\tau_A\right)_1 = \int_{S_\varepsilon}\left(y_1\frac{\partial (v_A)_2}{\partial y_3} - y_2\frac{\partial (v_A)_1}{\partial y_3}\right)dy_1 dy_2 + O(d^{1/2}).$$

We can see that y_1, y_2 are replaced by $-y_1, -y_2$ respectively, the value of q_A given by (21) is unchanged whereas $(v_A)_1, (v_A)_2$ given by (17) become $-(v_A)_1, -(v_A)_2$ respectively. Hence the force F_A, τ_A can be estimated by

$$(F_A)_1 = O(d^0),\ \ (F_A)_2 = O(d^0),\ \ (F_A)_3 = d^{-1}\int_{S_\varepsilon}(-q_A)dy_1 dy_2 + O(\ln d), \tag{22}$$

and

$$(\tau_A)_1 = O(d^0),\ (\tau_A)_2 = O(d^0),\ (\tau_A)_3 = d^{-1}\int_{S_\varepsilon}\left(y_1\frac{\partial(v_A)_2}{\partial y_3} - y_2\frac{\partial(v_A)_1}{\partial y_3}\right)dy_1dy_2 + O(d^{1/2}). \tag{23}$$

Substituting the formula of q_A given by (20), we have

$$(F_A)_3 = -d^{-1}\int_{\bar r=0}^{d^{-1/2}\varepsilon}\int_{\theta=0}^{2\pi}\bar r q_A d\bar r d\theta + O(\ln d) = 6\pi d^{-1}\int_{\bar r=0}^{d^{-1/2}\varepsilon}\bar r(1+\bar r^2)^{-2}d\bar r + O(\ln d).$$

Moreover, we have

$$\int_{\bar r=0}^{d^{-1/2}\varepsilon}\bar r(1+\bar r^2)^{-2}d\bar r = \frac{1}{2}\left(1-\frac{1}{1+d^{-1}\varepsilon}\right)\to\frac{1}{2},\ \text{as } t \text{ tends to } 0.$$

It implies $(F_A)_3 = 3\pi d^{-1} + O(\ln d)$.

Substituting $(v_A)_1$ and $(v_A)_2$ given by (17) into the expression of τ_A in (23), we get

$$(\tau_A)_3 = \int_{S_\varepsilon} h\left(y_2\frac{\partial q_A}{\partial y_1} - y_1\frac{\partial q_A}{\partial y_2}\right)dy_1dy_2 + O(d^0).$$

Using polar coordinates $y_1 = \bar r\cos\theta, y_2 = \bar r\sin\theta$, we obtain

$$(\tau_A)_3 = \frac{1}{2}\int_{\bar r=0}^{d^{-1/2}\varepsilon}\int_{\theta=0}^{2\pi}\bar r(1+\bar r^2)O(d)d\bar r d\theta + O(d^0) = f(\varepsilon) + O(d^0),$$

where $f(\varepsilon)$ tends to 0 as ε tends to 0. Therefore, we get $\left(\tau_A\right)_3 = O\left(d^0\right)$.

4.2. Tangential and rolling motion of spheres

Since (v_B, q_B) satisfies (7) and (8), we can do similar to the previous section, the value of the flow field (v_B, q_B) is given by:

$$q_B = q_B(y_1, y_2),\ (v_B)_1 = \frac{1}{2}\frac{\partial q_B}{\partial y_1}y_3^2 + Ay_3 + V_1,\ (v_B)_2 = \frac{1}{2}\frac{\partial q_B}{\partial y_2}y_3^2 + By_3 + V_2, \tag{24}$$

$$(v_B)_3 = -\frac{1}{6}\left(\frac{\partial^2 q_B}{\partial y_1^2} + \frac{\partial^2 q_B}{\partial y_2^2}\right)y_3^3 - \frac{1}{2}\left(\frac{\partial A}{\partial y_1} + \frac{\partial B}{\partial y_2} + y_1\frac{\partial q_B}{\partial y_1} + y_2\frac{\partial q_B}{\partial y_1}\right)y_3^2$$
$$-(Ay_1 + By_2)y_3 + \omega_1 y_2 - \omega_2 y_1, \tag{25}$$

where *A*, *B* are the functions of y_1, y_2 and are given by:

$$A = \frac{-2V_1}{h} - \frac{1}{2}\frac{\partial q_B}{\partial y_1}h, \qquad B = \frac{-2V_2}{h} - \frac{1}{2}\frac{\partial q_B}{\partial y_2}h.$$

Substituting the value of $\left(v_B\right)_3$ into the last boundary condition, we obtain

$$\nabla.\left(h^3\nabla q_B\right) = 24(\omega_2 y_1 - \omega_1 y_2). \tag{26}$$

We use the polar coordinates again, the above equation has the form

$$\bar{r}^2\frac{\partial^2 q_B}{\partial \bar{r}^2} + \frac{\partial^2 q_B}{\partial \theta^2} + \left(\bar{r} + \frac{6\bar{r}^3}{1+\bar{r}^2}\right)\frac{\partial q_B}{\partial \bar{r}} = 24\left(\omega_2\cos\theta - \omega_1\sin\theta\right)\frac{\bar{r}^3}{(1+\bar{r}^2)^3}. \tag{27}$$

Here we just need the asymptotic expansion of q_B for large $\bar{r}$, so we only require the form of q_B by using the limiting form of (27), we have

$$\bar{r}^2\frac{\partial^2 q_B}{\partial \bar{r}^2} + \frac{\partial^2 q_B}{\partial \theta^2} + 7\bar{r}\frac{\partial q_B}{\partial \bar{r}} = 24\left(\omega_2\cos\theta - \omega_1\sin\theta\right)\bar{r}^{-3}. \tag{28}$$

Similar to the case of q_A, we require q_B to satisfy

$$q_B = O\left(\bar{r}^{-3}\right) \quad \text{as} \quad \bar{r} \to +\infty. \tag{29}$$

The solution of (28) satisfy (29) is

$$q_B = -\frac{12}{5}\bar{r}^{-3}\left(\omega_2\cos\theta - \omega_1\sin\theta\right). \tag{30}$$

Due to the approximation in (28) that $\bar{r}$ was very large, (30) for q_B gives really the first term in the asymptotic expansion of q_B for large $\bar{r}$. Also since the expression in the boundary condition have an error of order *d*, so q_B is given by

$$q_B = -\frac{12}{5}\bar{r}^{-3}\left(\omega_2\cos\theta - \omega_1\sin\theta\right) + O\left(\bar{r}^{-2}\right) + O\left(d\right). \tag{31}$$

If we replace y_1, y_2 by $-y_1, -y_2$ respectively or equivalently θ by $\pi+\theta$, the value q_B becomes $-q_B$, while $\left(v_B\right)_1$ and $\left(v_B\right)_2$ are unchanged. Using these symmetric properties of the flow, the force F_B and torque τ_B on ∂B_1 generated by the flow (v_B, q_B) are calculated from (9) and (10) as

$$(F_B)_1 = \int_{S_\varepsilon}\left(-y_1 q_B + \frac{\partial\left(v_B\right)_1}{\partial y_3}\right)dy_1 dy_2 + O\left(d^{1/2}\right),$$

$$(F_B)_2 = \int_{S_\varepsilon}\left(-y_2 q_B + \frac{\partial\left(v_B\right)_2}{\partial y_3}\right)dy_1 dy_2 + O\left(d^{1/2}\right),$$

$$(F_B)_3 = O\left(d^0\right),$$

and

$$(\tau_B)_1 = \int_{S_\varepsilon}\left(-\frac{\partial\left(v_B\right)_2}{\partial y_3}\right)dy_1dy_2 + O\left(d^{1/2}\right),$$

$$(\tau_B)_2 = \int_{S_\varepsilon}\left(-\frac{\partial\left(v_B\right)_1}{\partial y_3}\right)dy_1dy_2 + O\left(d^{1/2}\right),$$

$$(\tau_B)_3 = O\left(d^0\right).$$

Substituting the values of $\left(v_B\right)_1$ and $\left(v_B\right)_2$ from (24) into these expressions, we obtain

$$(F_B)_1 = \int_{S_\varepsilon}\left(-y_1q_B + \frac{-2V_1}{h} - \frac{1}{2}h\frac{\partial q_B}{\partial y_1}\right)dy_1dy_2 + O\left(d^{1/2}\right),$$

$$(F_B)_2 = \int_{S_\varepsilon}\left(-y_2q_B + \frac{-2V_2}{h} - \frac{1}{2}h\frac{\partial q_B}{\partial y_2}\right)dy_1dy_2 + O\left(d^{1/2}\right).$$

These integrals can be evaluated by changing from $\left(y_1, y_2\right)$ to polar coordinates $\left(\bar{r},\theta\right)$ and by substituting the value of q_B from (31), the above expression for $\left(F_B\right)_1$ becomes

$$(F_B)_1 = \frac{\pi}{2}\int_0^{d^{-1/2}\varepsilon}\left(\frac{24\omega_2}{5}\bar{r}^{-1} - 8V_1\frac{\bar{r}}{1+\bar{r}^{2}} - \frac{24\omega_2}{5}\frac{1+\bar{r}^{2}}{\bar{r}^{3}}\right)d\bar{r} + O(d^0).$$

So we get the asymptotic form of $\left(F_B\right)_1$ as $\left(F_B\right)_1 = 2\pi V_1 \ln d + O(d^0)$.

By performing the similar computation for $\left(F_B\right)_1$, $\left(\tau_B\right)_1$ and $\left(\tau_B\right)_2$, we also obtain

$$\left(F_B\right)_2 = 2\pi V_2 \ln d + O(d^0),$$

and

$$\left(\tau_B\right)_1 = \left(-2\pi V_2 + \frac{6\pi}{5}\omega_1\right)\ln d + O(d^0),$$

$$\left(\tau_B\right)_2 = \left(-2\pi V_1 + \frac{6\pi}{5}\omega_2\right)\ln d + O(d^0).$$

4.3. Rotational motion of spheres

As in two previous sections, since (v_C, q_C) satisfies (7) and (8), we obtain

$$q_C = q_C(y_1, y_2), \quad \left(v_C\right)_1 = \frac{1}{2}\frac{\partial q_C}{\partial y_1}y_3^2 + Ay_3 - \omega_3 y_2, \quad \left(v_C\right)_2 = \frac{1}{2}\frac{\partial q_C}{\partial y_2}y_3^2 + By_3 - \omega_3 y_1,$$

$$\left(v_C\right)_3 = -\frac{1}{6}\left(\frac{\partial^2 q_C}{\partial y_1^2} + \frac{\partial^2 q_C}{\partial y_2^2}\right) y_3^3 - \frac{1}{2}\left(\frac{\partial A}{\partial y_1} + \frac{\partial B}{\partial y_2} + y_1 \frac{\partial q_C}{\partial y_1} + y_2 \frac{\partial q_C}{\partial y_2}\right) y_3^2 - (Ay_1 + By_2) y_3.$$

where *A*, *B* are the functions of y_1, y_2 and are given by:

$$A = \frac{2\omega_3}{h} y_2 - \frac{1}{2}\frac{\partial q_C}{\partial y_1} h, \quad B = \frac{-2\omega_3}{h} y_1 - \frac{1}{2}\frac{\partial q_C}{\partial y_2} h.$$

Using the last boundary condition on $\left(v_C\right)_3$, we get an equation for q_C as follows

$$\nabla.(h^3 \nabla q_C) = 0.$$

This implies that $q_C = O(d)$. Hence, we can see that the force F_C and torque τ_C are no longer singular being of order d^0, it means $F_C = O(d^0)$, $\tau_C = O(d^0)$.

Finally, the proof of the main result is complete by combining with all the results in section 4.1, 4.2 and 4.3. So we obtain the asymptotic formulas of the force F and torque τ on ∂B_1 as claimed.

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