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AN EXPERIMENTAL STUDY OF THE DIFFICULTIES INVOLVED IN LEARNING THE GROUP CONCEPT

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ABSTRACT

In this paper, we present the experimental results in verifying the research hypothesis represented in the article "An epistemological analysis of the concept of quotient group" (Nguyen Ai Quoc, 2018) about the existence of three difficulties to students in first approaching the concept of quotient group. And these difficulties are derived from three epistemological obstacles: intrinsic, abstraction, and structuralisation.

Keywords: epistemological obstacles, difficulties, quotient groups.

TÓM TẮT

Một nghiên cứu thực nghiệm về các khó khăn liên quan đến việc học khái niệm nhóm thương

Trong bài báo này, chúng tôi trình bày kết quả thực nghiệm kiểm chứng giả thuyết nghiên cứu nêu trong bài báo "Một phân tích tri thức luận về nhóm thương" *(Nguyễn Ái Quốc, 2018) về sự tồn tại của ba khó khăn đối với sinh viên khi lần đầu tiên tiếp cận khái niệm nhóm thương và các khó khăn này có nguồn gốc từ ba chướng ngại tri thức luận của khái niệm nhóm thương: nội tại, trừu tượng hóa, và cấu trúc hóa.*

Từ khóa: chướng ngại tri thức luận, khó khăn, nhóm thương.

1. Introduction

1.1. Some preparable knowledge

The group concept is one of the abstract concepts of group theory. The definition of quotient group we refer to in this paper is the modern "standard definition" used in current curriculums. In the following, we present a brief description of the concept of quotient group and the concepts needed to build this concept, based on the syllabus "General Algebra" by Hoang Xuan Sinh, Tran Phuong Dung (2003).

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- ***Concept of coset***

The concept of cosets was introduced in the syllabus by Hoang Xuan Sinh, Tran Phuong Dung (2003) after the definition of equivalence relation in a group. The concept of left class and of right class must be defined as follows: "*The components xA are called the left classes of the subgroup A (or the left cosets mod A) in X. Similarly, the right classes Ax of A (or the right cosets mod A) in X are the components whose elements have the form ax with* $a \in A$" (p. 29)

- ***Concept of subgroup***

After reviewing the group structure and stability of the group, Hoang Xuan Sinh, Tran Phuong Dung (2003) defines the concept of subgroup as follows: "*A stable part A of a group X is called the subgroup of X if A with the induced operation makes a group*." (p. 22)

- ***Concept of normal group***

To bring about the concept of normal subgroups, Hoang Xuan Sinh, Tran Phuong Dung (2003, p. 31) addresses some of the constraints related to normalized subgroups such as:

- Equivalence relation on a subset;
- Concepts of left cosets and of right coset.

The concept of a normal subgroup is defined as follows:

"*A subgroup A of a group X is called normal subgroup if and only if* $x^{-1}ax \in A$ *for every* $a \in A$ *and* $x \in X$." (p. 31)

In addition, the concept of normal subgroup is equivalent to the following (characteristic) theorem of normal subgroup: "*Suppose A is a subgroup of a group X, the following conditions are equivalent:*

a) A is normal subgroup;

b) $xA = Ax$ *pour tout* $x \in X$*."* (p. 32)

- ***Concept of quotient group***

After introducing the concept of normal subgroups, Hoang Xuan Sinh, Tran Phuong Dung (2003, p. 32) defines quotient groups by the following theorem:

"*If A is a normal subgroup of a group X, then*:

(i) The rule for correspondence pair (xA, yA), the left class xyA is a mapping from X/A $\times$ *X/A to X/A;*

(ii) X/A with binary operations (xA, yA) $\rightarrow$ xyA makes a group, called the quotient group of X on A." (p. 32)

1.2. Some remarks and hypothesis

Presenting the concept of a quotient group through the concept of normal subgroups and the structuralization of the quotient group into classes by equivalence relation (section 1.1) can lead to difficulties faced by students (SV) when approaching the concept of quotient group. Indeed, in July 2016, Nguyen Ai Quoc (2018) conducted an initial interview among 8 students majored in Pedagogy of Mathematics in Saigon University and Dong Nai University on the concept of quotient group. The survey results have shown the three main difficulties during the interviews: the distinction between the elements of the quotient group and those of the original group, the comprehension of nature of the elements and that of the operations of quotient group, and the realization of fundamental factors in the constructing of a quotient group.

On the other hand, in the analysis of historical epistemology, Nguyen Ai Quoc (2018) identifies three important characteristics, appearing continuously and throughout the process of constructing the concept of quotient group:

- *Intrinsic* characteristic: the quotient group is derived from the elements of the original group;
- *Abstract* characteristic: associated with the construction of general quotient group;
- *Structural* characteristic: the quotient groups include equivalence relation, quotient set, cosets, equivalence classes, normal subgroup, homomorphism, isomorphism.

From there, Nguyen Ai Quoc (2018) identified three obstacles for students when first approaching the concept of quotient group:

- *Intrinsic obstacle*: associated with the construction of the quotient group from elements of the original group;
- *Abstract obstacle*: This obstacle generates the difficulties that students face when they transfer from research on sets of specific numbers (represented objects) to research on symbol systems (representative objects);
- *Structural obstacle*: associated with the structuralization of the quotient group into classes by equivalence relation.

These empirical and epistemological analysis allow Nguyen Ai Quoc (2018) to refer to the hypothesis **H** on the existence of three difficulties (DF) in most students when first approaching the concept of quotient group:

DF1: the distinction between the elements of the quotient group and those of the original group;

DF2: the comprehension of the nature of the elements and that of the operations of quotient group;

DF3: the mastery of fundamental factors in the buiding of a quotient group.

This hypothesis will be verified by an experimentation that we will present in this paper.

2. Experimentation

The experimentation were conducted among 143 students of three universities: Ho Chi Minh City Pedagogic University - training high school teachers, Sai Gon University - multidisciplinary university training secondary and high school teachers, and Dong Nai University - multidisciplinary and applied university. These students have completed the course in General Algebra, which includes knowledge of the quotient group. The students on whom we conducted the survey in July 2016 (section 1.2) will not participate in this experimentation.

The experimentation included a written questionnaire, which was conducted in June 2017, for a period of 45 minutes.

2.1. Experimental content

The questionnaire was designed to validate the hypothesis H on the three epistemological obstacles of the quotient group: intrinsic obstacle, abstract obstacle, and structural obstacle. These obstacles will be identified by determining three types of difficulties for students when approaching the concept of quotient group presented in section 1.2: distinguish between the elements of the quotient group and those of the original group; comprehend the nature of the elements and of the operations of quotient group; realize the fundamental factors in the buiding of a quotient group. Therefore, the questionnaire was designed to include the following types of tasks:

T1: Describe elements of a quotient group;

T2: Describe the relationship between the quotient group G/H and the original group G;

T3: Describe the normal subset H of group G;

T4: Describe the nature of the elements and of the operation in G/H;

T5: Describe the conditions for constructing a quotient group G/H from the original group G using the normal subgroup H of G.

The questionnaire consists of 6 questions as follows:

Question 1. Please tell each of the following statements True (S) or False (S)? *Given H as a subgroup of group G. H is normal subgroup of G if and only if ...*

$(a)\ \forall h \in H, \forall x \in G, \exists y \in H: hx = xy$

$(b)\ \forall h \in H, \forall x \in G, hx = xh$

$(c)\ \forall x \in H, \forall h \in H, xh = hx$

$(d)\ \forall h \in H, \forall x \in G, \exists y \in G: hx = xy.$

Question 1 is of task type T3: Describe the normal subset *H* of group *G*.

The purpose of question 1 is to examine whether the students recognize a definition of the normal subgroup in the proposition (*a*), which differentiates the definition of normal subgroup from the definition of group's center in the proposition (*b*), and from the definition of commutative subgroup in the proposition (*c*), or from the statement applied to any subgroup in the proposition (*d*).

Question 2. *To construct a quotient group G/H, is H necessarily a normal subgroup of G? Please explain your answer.*

Question 2 is of task type T5: Describe the conditions for constructing a quotient group G/H from the original group G using the normal subgroup H of G.

The purpose of question 2 is to determine if the students know that *H* must be a normal subgroup to construct the quotient group *G/H* and see if they can explain it.

The correct answer is "yes" because the quotient group is constructed by the group *G* and the normal subgroup *H*. To be more exact, the normal condition of *H* is necessary to be able to construct a binary operation on the quotient set *G/H* to make it a quotient group.

Question 3. *Is the quotient group G/H always a subgroup of the group G? Please explain your answer.*

Question 3 is of task type T2: Describe the relationship between the quotient group G/H and the original group G.

The purpose of question 3 is to find out if there is any idea that G/H is (or could be) a subgroup of G, and if so, to learn why students answer "yes" to this question.

The correct answer is "no" because G/H is not a subset of G and therefore it can not be a subgroup of G, although if G is cyclic, then G/H is isomorphic to a subgroup of G.

Question 4. *Can the elements of the quotient group G/H be elements of G? Please explain your answer.*

Question 4 is of task type T1: Describe elements of a quotient group.

The purpose of question 4 is to determine if there is any idea that the elements of the quotient group G/H are (or may be) elements of G, and, if so, to learn why the students answer "yes" to this question.

The correct answer is "no" because the element of quotient group G/H is a coset following the normal subgroup H so it can not be an element of G.

Question 5. *Can we establish one (or many) relationships between the phrase "quotient group" and the meaning of the word "quotient" in arithmetic? Explain why?*

Question 5 is of task type T4: Describe the nature of the elements and of the operation in G/H.

The purpose of question 5 is to determine whether or not the incorrect relationship is established between the quotient group and the arithmetic quotient by the students in answering this question. More specifically, determine if there is any idea that each element of the quotient group is the ratio of one element of G to one element of H.

The correct answer is "yes" because the quotient group is the generalization of the set of congruence classes in arithmetic for an arbitrary group.

Question 6. *Are the elements of a quotient group G/H the equivalence classes? If you answer "Yes", please tell what the relative equivalence relation is?*

Question 6 is of task type T1: Describe elements of a quotient group.

The purpose of question 6 is to determine whether the students understand that the elements of a quotient group are equivalence classes, and whether in the case of a "yes" answer, they can describe the relative equivalence relation.

The correct answer is "yes" because an element of the quotient group G/H is an equivalence class. The relative equivalence relation is that "$\forall a, b \in G, a \sim b \Leftrightarrow aH = bH \Leftrightarrow \exists h \in H, a = bh$."

2.2. Anticipation of students' difficulties when answering questions

Table 1 below summarizes the task types associated with the above six questions and the ability to determinine the students' difficulties.

Table 1. *Difficulties associated with answering each empirical question*

Questions	Task types	Difficulties
Q1	T3: Describe the normal subset H of group G	DF3
Q2	T5: Describe the conditions for constructing a quotient group G/H from the original group G using the normal subgroup H of G	DF2, DF3
Q3	T2: Describe the relationship between the quotient group G/H and the original group G	DF1, DF2
Q4	T1: Describe elements of a quotient group	DF1, DF2
Q5	T4: Describe the nature of the elements and of the operation in G/H.	DF2, DF3
Q6	T1: Describe elements of a quotient group	DF2, DF3

2.3. Posterior analysis

In this section, we present the results of the analysis of the students' responses to each empirical question

Question 1. Please tell each of the following statements True (S) or False (S)?

Given H as a subgroup of group G. H is normal subgroup of G if and only if ...

$(a)\ \forall h \in H, \forall x \in G, \exists y \in H: hx = xy$

$(b)\ \forall h \in H, \forall x \in G, hx = xh$

$(c)\ \forall x \in H, \forall h \in H, xh = hx$

(d) $\forall h \in H, \forall x \in G, \exists y \in G: hx = xy.$

Among 143 students who answered the questionnaire, 78 persons (54.55%) did not choose the proposition (a) (i.e. that the proposition (a) was wrong), that means they did not recognize this as a statement that allows to define a normal subgroup.

Among 62 students (43.36%) who chose the proposition (a) (i.e. that the proposition (a) was correct), 11 also chose the proposition (b) defining the group center and 6 also chose the proposition (c) defining commutative subgroups, i.e., they think that these propositions are equivalent to the proposition (a).

Three students did not answer question 1, which means they could not percieve the four propositions as correct or wrong. Among 78 students who did not choose the proposition (a), 35 persons (17.48%) chose the proposition (b) defining the group center (including 11 persons who also chose the proposition (a) above), 27 persons (18.88%) chose the proposition (c) defining a commutative subgroup (including the 6 persons who also chose the proposition (*a*) above) and 16 persons (11.19%) chose the proposition (*d*) that is a statement applied to an arbitrary subgroup.

These results show that students have difficulty in describing normal subgroups. This difficulty belongs to type DF3: “the realization of fundamental factors in the building of a quotient group”.

Table 2. *Response results of the question 1 of students*

Questions	1a (True)	1a (False)	1b (True)	1c (True)	1d (True)
Amount	62/143	78/143	35/143	27/143	16/143
Percentage	43,36%	54,55%	17,48%	18,88%	11,19%

Question 2. *To construct a quotient group G/H, is H necessarily a normal subgroup of G? Please explain your answer*

33 students (23.08%) answer “no” to question 2, which means they do not think that subgroup H is not necessarily the normal subgroup to build a quotient group G/H.

On the other hand, among 110 respondents (76.92%) who answered “yes”, 83 did not adequately explain their answers. For example, 41 students explained that “H is necessarily a normal subgroup because the quotient group must be built on the subgroup of the original group G” or "according to the definition of quotient group G/H, H must be a normal subgroup".

The results show that the majority of students (76.92%) know that to build a quotient group G/H, subgroup H must be a normal subgroup, but most (75.45%) of this group can not explain why. Thus, the majority of students met the difficulty of type DF2: “the comprehension of nature of the elements and that of the operations of quotient group”, and DF3: “the realization of fundamental factors in the building of a quotient group”.

Question 3. *Is the quotient group G/H always a subgroup of the group G? Please explain your answer*

121 students answered “yes” to question 3, which means they considered the quotient group G/H a subgroup of group G. Among them, 24 students explained that because the quotient group is constructed from the normal subgroup H of G, the quotient group G/H is a subgroup of G; 87 students explained that the elements of the quotient group G/H are elements of the original group G; there are 10 students who explained that if H = <e> then G/H is G.

Analytical results show that students met the difficulty of type DF1: “the distinction between the elements of the quotient group and those of the original group”, and of type DF2: “the comprehension of the nature of the elements and that of the operations of quotient group". These difficulties are related to the conception that the quotient group is made up of the normal subgroup H and from the elements of the original group G.

Question 4. *Can the elements of the quotient group G/H be elements of G? Please explain your answer*

Similar to the result of question 3, in question 4, there are 121 "yes" answers, which means that the elements of G/H can be elements of G. Among these answers, there are many similar answers as: "The element of G/H is the product of an element of H and an element of G, because H is a subgroup of G, that element should belong to G", or "Because elements of G/H are equivalence classes consisting of many elements of G, they should belong to G", or "Because the quotient group G/H is constructed by elements of G, the elements of G/H should belong to G", or "Because the quotient group G/H is a set of the equivalence classes of G with normal subgroup H, if *x* belongs to G/H, then *x* should belong to an equivalence class of G with a normal subgroup of G, thus x should belong to G".

Among 22 correct answers to question 4, most explain that because an element of the quotient group G/H is an equivalence class, it can not be an element of G.

Thus, when answering question 4, students have difficulty of type DF1: "the distinction between the elements of the quotient group and those of the original group", and of type DF2: "the comprehension of the nature of the elements and that of the operations of quotient group". These difficulties are related to the conception that the quotient group G/H is made up of the elements of the original group G.

Question 5. *Can we establish one (or many) relationships between the phrase "quotient group" and the meaning of the word "quotient" in arithmetic? Explain why?*

For Question 5, 89 students answered "no", which means that they could not establish the relationship between the phrase "quotient group" and the meaning of word "quotient" in arithmetic, but most did not adequately explain the answer. For example, there are such explanations as: "The quotient group is the phrase associated with the symbol G/H of the quotient group, unlike the quotient of two numbers in the arithmetic", or "The quotient group can not be compared to the set of numbers".

However, 37 students answered "yes" and explained that "there is a similarity in partitioning of a group into cosets to create the quotient group and partitioning of a set of numbers into subsets by congruence relations", or "the division in quotient groups is similar to the division in arithmetic". The remaining 17 students answered "do not know". In particular, none of the students mentioned the division of each element of G by each element of H as found in the original survey experiment.

Thus, the answers of the students showed that they met difficulties of type DF2: "the comprehension of nature of the elements and that of the operations of quotient group", or of type DF3: "the realization of fundamental factors in the building of a quotient group".

Question 6. *Do the elements of a quotient group G/H have to be the equivalence classes? If you answer "Yes", please tell what the equivalence is?*

123 students answered "yes", which means they know that the elements of the quotient group G/H are equivalence classes. However, only 55 students have written out the equivalence relation, and the rest are either incorrect or incomplete.

Thus, although most students know that the elements of the quotient group are equivalence classes, they still meet difficulties of type DF3: "The realization of fundamental factors in the building of a quotient group".

3. Conclusion

The experimental results show the existence of three types of difficulties the students face when they first approach the concept of quotient groups: the distinction between the elements of the quotient group and those of the original group (DF1), the comprehension of the nature of the elements and that of the operations of quotient group (DF2), and the realization of fundamental factors in the building of a quotient group (DF3).

The first type of difficulty (DF1) is evident in the students' answers to questions 3 and 4, in which they perceive that the quotient group G/H is a subgroup of the original group G and the elements of the quotient group are the element of the original group G. This difficulty comes from the intrinsic obstacle associated with the construction of the quotient group from the original group.

The second type of difficulty (DF2) is shown by the students' answers to questions 2, 3, 4, 5 and 6, in which they do not have a clear conception of the shape of the elements of the quotient group and how the operation is performed. This type of difficulty comes from the abstract obstacle as students transfer from research on specific sets of numbers (represented objects) to research on symbol systems (representative objects) and from the obstacle associated with the structuralization of the quotient group from the original group into equivalence classes.

The third type of difficulty (DF3) is shown by the students' answers to questions 1 and 2, in which they neither recognize the structure of a normal subgroup nor explain why the subgroup H must be group normal subgroup.

The experimental results allow us to validate the hypothesis H of the existence of three types of difficulties that are derived from three obstacles: intrinsic, abstract, and structural.

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